Advanced Dynamics

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Foreword

Advanced Dynamics by Shuh-Jing (Benjamin) Ying provides a comprehensive introduction to this important topic for aeronautical or mechanical engineering students. It is written with the student in mind by explaining in great detail the fundamental principles and applications of advanced dynamics. The applications are first illustrated on simple problems, such as the collision of two bodies, and then demonstrated on much more complex problems, such as a two-impulse trajectory for space probes. Dr. Ying is a Professor at the University of South Florida in the Department of Mechanical Engineering, and his research interests include dynamics, vibrations, mechanical design, and heat transfer. Also, in addition to his extensive research activity and numerous publications, Dr. Ying has taught 34 different courses in mechanical engineering.

The text covers all the essential mathematical tools needed to analyze the dynamics of systems: vector algebra, conversion of coordinates, calculus of variations, matrix algebra, Cartesian tensors and dyadics, rotation operators, Fourier series, Fourier integrals, Fourier transforms, and Laplace transforms (in Chapters 1, 6, and 8). Chapters 1 through 3 start with a review of elementary statics and dynamics, followed by a discussion of Newton's laws of motion, D'Alembert's principle, virtual work, and kinematics and dynamics of a single particle or system of particles. Chapter 4 introduces Lagrange's equations and the variational principle used in dynamics. Chapter 5 is devoted to the dynamics of rockets and space vehicles, while Chapters 7, 8, and 9 discuss the dynamics of a rigid body and vibrations of continuous systems as well as lumped parameter systems with a single degree or multiple degrees of freedom. Nonlinear vibrations are also included. Chapter 10 discusses the Special Theory of Relativity and its consequences in kinematics and dynamics.

The Education Series of textbooks and monographs published by the American Institute of Aeronautics and Astronautics embraces a broad spectrum of theory and application of different disciplines in aeronautics and astronautics, including aerospace design practice. The series also includes texts on defense science, engineering, and management. Over 50 titles are now included in the series, and the books serve as both teaching texts for students and reference materials for practicing engineers, scientists, and managers. This recent addition to the series will be a valuable text for courses in engineering dynamics in aeronautical or mechanical engineering programs.

J. S. Przemieniecki
Editor-in-Chief
AIAA Education Series

Preface

Dynamics is the foundation of physical science and is an important subject of study for all engineering students. Although the fundamental laws of dynamics have remained unchanged, their applications are constantly changing. One hundred years ago, there were no automobiles, no airplanes, and no space vehicles. Advances in science and technology provide us with many new dynamic devices. For example, the gyroscopic effect of the rotating propeller in airplanes creates diving during yawing. When a satellite travels in a circular orbit, the motions of rolling and yawing also can produce pitching. During times of war, shooting a missile flying in its orbit is another subject with real and important implications. Is it possible to shoot a space probe from the surface of Earth to Mars by one impulse? All these scenarios are important and interesting, and understanding them begins with the study of dynamics.

As I teach advanced dynamics, I feel that there is a need for a textbook that covers subjects related to recent developments. A book that includes my lecture notes may fulfill this need, and this is my primary motivation for writing this book. In addition, this book is intended not only for students in the classroom but also for practicing engineers who wish to update their knowledge. For this reason, the book is self-contained with fundamentals in vector algebra, vector analyses, matrix operations, tensors and dyadics. The details are clearly and explicitly presented. I have been teaching advanced dynamics for more than 10 years, and I often tell my students that I have nothing to hide. This is the spirit of this book. Anyone who reads the book should not only understand current developments in dynamics but also can learn some of the foundations of mechanical engineering necessary to understand papers published in recent journals.

Further, I hope this book will show the reader that dynamics is an exciting field with many new problems to be solved. For example, there are challenging problems concerning the motion of a space vehicle traveling in a general orbit, and also in the design of robots and complex automatic machines. Lastly, a chapter on the special relativity theory is included. This is intended to show that space and time are related. Just a few days in one system can be many years in the other system. Past events in the stationary system can be observed at present in another system traveling near the speed of light. All these are not fairy tales, but are scientifically true. The purpose of this part of the book is to broaden readers' minds. Anything is possible.

The contents of this book are briefly described as follows: In Chapter 1, fundamental principles and vector algebra are reviewed. This chapter may be skipped by well-prepared students. Chapter 2 deals with kinematics and dynamics of a particle. First, the kinematics of a particle in various coordinate systems is discussed. Next, examples concerning trajectories of missiles and reentry of space vehicles are presented. Lastly, fundamental concepts such as work, conservative force, and potential energy are reviewed. Chapter 3 is devoted to the dynamics of a system of particles. Besides items commonly introduced in this chapter, the mid-air collision of missiles is given in detail including a computer program that determines the trajectory of the second missile. Collisions of solid spheres are also introduced in this chapter. This can be considered as the first approximation for automobile collisions. To balance theoretical aspects and practical applications, gravitational force and potential energy also are studied in this chapter.

Chapter 4 is a major chapter in this book. Many important topics are included. Many engineering students have difficulty formulating equations for motion for a particle or a body. Lagrange's equation is intended to help students find the equation of motion. Students only need to have the knowledge of kinetic and potential energies of the mass for formulating the equations. Hamilton's principle is a parallel approach to Lagrange's equations. With the study of Hamilton's principle, students will better understand the equations of motion. Lagrange's equations with constraints also are introduced. Constraint forces and Lagrange multipliers are derived. Many examples are given for Lagrange's equations. Students should be familiar with this subject if a proper effort is devoted to study. The variational principle is included in this chapter. Through this approach, Lagrange's equation for a conservative system also can be reached. The purpose of the variational principle is for optimization. A case of optimization with a constraint condition is studied also. Many examples are given to demonstrate the application of the variational principle.

Chapter 5 is devoted to the dynamics of rockets and space vehicles. This is another demonstration of the balance of theory and practice in this book. Essential characteristics of rockets are studied in a single-stage rocket. The advantage of multistage rocket and use of the Lagrangian multiplier for maximizing the burnout velocity are included. A space vehicle traveling in a gravitational field is treated extensively in Section 5.3. Different trajectories are discussed. Special attention is devoted to the elliptical orbit. The trajectory for an electrical-propulsion rocket is given in Section 5.4. The equations involved in electrical propulsion typically belong to a small perturbation theory. Equations of motion are solved analytically in the chapter. Interplanetary trajectories are discussed in Section 5.5. The journey from Earth to Mars' surface is used to demonstrate the procedure for calculating the impulses required for the whole trajectory. After a review of previous work, the use of two impulses for sending a space probe from Earth to Mars' orbit and spiraling down to the surface of Mars is discussed in detail. In this way the long and detailed observations can be made by the space probe.

Chapter 6 is for matrices, tensors, dyadics, and rotation operators. This chapter is entirely mathematical, so that engineering students are exposed to more applied mathematics. Some applications are included with each subject to make them easily understandable and more interesting. For example, through rotation operators it is proved that two successive rotations can be combined into a single rotation. This can actually reduce the time for rotational motions. Engineers wishing to extend their knowledge through journal papers should pay special attention to this chapter.

The dynamics of a rigid body are studied in Chapter 7. Because many objects may be modeled as rigid bodies, the analyses presented in this chapter play an important role in this book. The first three sections present fundamental principles. Some additional sections are included here describing the gyroscope and the orbiting space vehicle. The gyroscopic effect of a rotating propeller in an airplane causing the plane to dive during yawing is studied here in detail. The major application of the angular momentum of a rigid body is the gyro-compass. Two examples are particularly aimed in that direction. Furthermore, the motion of a heavy symmetrical top and induced torques because of flight operations on a satellite in circular orbit also are treated in detail in this chapter.

Chapters 8 and 9 are devoted to the study of vibrations. In Chapter 8, mathematical topics that are necessary for analyzing vibration problems are first presented. These topics are Fourier series, Fourier integral, and Fourier and Laplace transforms. The Laplace transform is treated as a special case in Fourier transformation. Applications include one-dimensional damped oscillations and transient vibrations. Advanced topics in vibration are treated in Chapter 9. Starting from a two-degree-of-freedom system, some examples in a lumped parameter system, a continuous system, and nonlinear vibrations are studied. Stability analysis of vibrations in a phase plane is also discussed.

Chapter 10 covers the Special Relativity Theory. This is arranged here to broaden readers' minds. The time and space coordinates are related such that for one person traveling near the speed of light, just a few days for this person can be many years to a person in a stationary system. This is proved to be true scientifically. Moreover one also can prove that an event in the past could be observed as a present event in another system. Readers are urged to consider that, just as space and time are now interrelated through the relativity theory, new developments may one day modify our thoughts concerning our most basic scientific concepts and principles.

In conclusion, I wish to thank Sue Britten for providing valuable support in the process of accomplishing this book.

Shuh-Jing (Benjamin) Ying
July 1997

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1
Review of Fundamental Principles

THIS chapter reviews the fundamental principles necessary for the study of advanced dynamics. Although these principles may be familiar to students who have studied elementary mechanics, they are included here so that this book is reasonably self-contained.

The concepts of dimensions and units are reviewed in Section 1.1. Familiarity with these concepts will greatly facilitate formulating equations, checking dimensional homogeneity of an equation, and converting units. A brief review of vector analysis is given in Section 1.2. Formulas frequently used in this book are presented. Section 1.3 contains the definitions of statics and dynamics and a discussion of the difference between kinematics and kinetics. Section 1.4 presents Newton's laws of motion. The second law is written in an expanded form to include the effect of changing mass, which is essential for analyzing the dynamics of a rocket or any object with variable mass. D'Alembert's principle is presented in Section 1.5. Through the use of D'Alembert's principle, dynamic problems are simplified to static ones. Section 1.6 reviews the principles of virtual displacement and virtual work, which are the foundation for the derivation of Lagrange's equations discussed in Chapter 4.

1.1 Dimensions and Units

A dimension is the measure by which the magnitude of a physical quantity is expressed. In dynamics, there are usually four dimensions: mass, length, time, and force. A unit is a determinate quantity adopted as a standard of measurement. As shown in Table 1.1, the International System of Units (SI) specifies mass in kilograms (kg), length in meters (m), time in seconds (s), and force in newtons (N). In the British Gravitational System (BG), mass is measured in slugs, length in feet (ft), time in seconds (s), and force in pounds (lbf). It is important to mention that understanding dimensions and units will prevent errors from occurring when analyzing problems and converting units. The conversion factors for the two systems are given in Table 1.1.

Of the four dimensions mentioned in Table 1.1, mass, length, and time are considered as primary dimensions and force as a secondary dimension. Force can be expressed in terms of mass, length, and time as follows:

$$1\ \text{N} = 1\ \text{kg-m/s}^2 \tag{1.1}$$

$$1\ \text{lbf} = 1\ \text{slug ft/s}^2 \tag{1.2}$$

The following example illustrates the technique used in the conversion of units.

$$\begin{aligned} 1\ \text{km/s} &= 1000\frac{\text{m}}{\text{s}}\ \frac{1\ \text{ft}}{0.3048\ \text{m}}\ \frac{1\ \text{mile}}{5280\ \text{ft}}\ \frac{3600\ \text{s}}{1\ \text{h}} \\ &= 2236.94\ \text{mph} \end{aligned}$$

Table 1.1 Conversion factors

Dimensions	SI unit	BG unit	Conversion factor
Mass, M	Kilogram, kg	Slug	1 slug = 14.5939 kg
Length, L	Meter, m	Foot, ft	1 ft = 0.3048 m
Time, T	Second, s	Second, s	1 s = 1 s
Force, F	Newton, N	Pound, lbf	1 lbf = 4.4482 N

When discussing units and dimensions, it is worthwhile to mention that each term in an equation must have the same dimension, and the dimensions on both sides of the equal sign must be the same. This is known as the principle of dimensional homogeneity. Application of this principle will prevent algebraic errors from occurring in complicated manipulations of equations.

1.2 Elements of Vector Analysis

Physical quantities in mechanics can be expressed mathematically by means of scalars and vectors. A quantity characterized by magnitude only is called a scalar. Mass, length, time, and volume are scalar quantities. A vector is a quantity that has both a magnitude and direction and obeys the parallelagram law of addition. Force, velocity, acceleration, and position of a particle in space are vector quantities.

A vector can be broken down into several components according to convenience. In the Cartesian coordinate system, a vector $\boldsymbol{a}$ can be expressed in its components as

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$$

where a_x, a_y, and a_z are the components of the vector, and $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the corresponding unit vectors. Because vector analysis plays an important role in dynamics, fundamental mathematics of vectors is presented in this section. Note that throughout the book, vectors are denoted by bold letters.

Vector Algebra

Vector addition. The addition of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is computed as

$$\begin{aligned} \boldsymbol{c} &= \boldsymbol{a} + \boldsymbol{b} \\ &= a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k} + b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k} \\ &= (a_x + b_x)\boldsymbol{i} + (a_y + b_y)\boldsymbol{j} + (a_z + b_z)\boldsymbol{k} \end{aligned} \tag{1.3}$$

Vector subtraction. Vector subtraction, being a special case of vector addition, is performed as

$$\begin{aligned} \boldsymbol{c} &= \boldsymbol{a} - \boldsymbol{b} \\ &= (a_x - b_x)\boldsymbol{i} + (a_y - b_y)\boldsymbol{j} + (a_z - b_z)\boldsymbol{k} \end{aligned} \tag{1.4}$$

Scalar product of two vectors. The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written as $\boldsymbol{a} \cdot \boldsymbol{b}$, which is a scalar quantity, and is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab\cos\theta \tag{1.5}$$

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{ab} = \frac{a_x b_x + a_y b_y + a_z b_z}{ab} \tag{1.6}$$

where a_x, a_y, a_z, b_x, b_y, and b_z are components of vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and a is the magnitude of vector $\boldsymbol{a}$ and b the magnitude of vector $\boldsymbol{b}$.

Cross product of two vectors. The cross product of two vectors is written as $\boldsymbol{a} \times \boldsymbol{b}$, which is a vector, and is defined as

$$\boldsymbol{a} \times \boldsymbol{b} = (ab\sin\theta)\boldsymbol{e}$$

where θ is the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{e}$ is a unit vector perpendicular to the plane containing vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and in the direction according to right-hand rule. The mathematical operation of the cross product is performed as follows:

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\ &= \boldsymbol{i}(a_y b_z - a_z b_y) + \boldsymbol{j}(a_z b_x - a_x b_z) + \boldsymbol{k}(a_x b_y - a_y b_x)\end{aligned} \tag{1.7}$$

Triple scalar product. The triple scalar product of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is defined as $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$. The result is a scalar quantity and is obtained as

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \tag{1.8}$$

Triple vector product. The triple vector product of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is defined as $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$. The result is a vector quantity and is obtained as

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{a})\boldsymbol{c} \tag{1.9}$$

Differentiation

The derivative of a vector, which is a function of time, is defined as

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{V}(t + \Delta t) - \boldsymbol{V}(t)}{\Delta t} \tag{1.10}$$

From the definition given in Eq. (1.10), the derivatives of the product of a scalar and vector, the scalar product of two vectors, and the cross product of two vectors are given in the following equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha \boldsymbol{V}) = \frac{\mathrm{d}\alpha}{\mathrm{d}t}\boldsymbol{V} + \alpha\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \tag{1.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\cdot\boldsymbol{b}) = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} \tag{1.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\times\boldsymbol{b}) = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\times\boldsymbol{b} + \boldsymbol{a}\times\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} \tag{1.13}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{V}$ are vectors and α is a scalar. If $\boldsymbol{V}$ is expressed in its Cartesian components, then $\boldsymbol{V} = V_1\boldsymbol{i} + V_2\boldsymbol{j} + V_3\boldsymbol{k}$, and its derivative is

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \frac{\mathrm{d}V_1}{\mathrm{d}t}\boldsymbol{i} + \frac{\mathrm{d}V_2}{\mathrm{d}t}\boldsymbol{j} + \frac{\mathrm{d}V_3}{\mathrm{d}t}\boldsymbol{k} \tag{1.14}$$

In a general case, the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ may change their orientations in space as time progresses; then $\boldsymbol{V} = V_1\boldsymbol{e}_1 + V_2\boldsymbol{e}_2 + V_3\boldsymbol{e}_3$, and the derivative of $\boldsymbol{V}$ can be written as

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \frac{\mathrm{d}V_1}{\mathrm{d}t}\boldsymbol{e}_1 + \frac{\mathrm{d}V_2}{\mathrm{d}t}\boldsymbol{e}_2 + \frac{\mathrm{d}V_3}{\mathrm{d}t}\boldsymbol{e}_3 + V_1\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} + V_2\frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t} + V_3\frac{\mathrm{d}\boldsymbol{e}_3}{\mathrm{d}t} \tag{1.15}$$

or

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \dot{V}_1\boldsymbol{e}_1 + \dot{V}_2\boldsymbol{e}_2 + \dot{V}_3\boldsymbol{e}_3 + V_1\dot{\boldsymbol{e}}_1 + V_2\dot{\boldsymbol{e}}_2 + V_3\dot{\boldsymbol{e}}_3$$

where $\dot{V}_i$ and $\dot{\boldsymbol{e}}_i$ are the time derivatives.

Gradient, Divergence, and Curl Operations

The gradient of a scalar ϕ is defined as

$$\text{Gradient } \phi = \nabla\phi = \boldsymbol{i}\frac{\partial\phi}{\partial x} + \boldsymbol{j}\frac{\partial\phi}{\partial y} + \boldsymbol{k}\frac{\partial\phi}{\partial z} \tag{1.16}$$

The divergence of a vector $\boldsymbol{F}$

$$\text{Div } \boldsymbol{F} = \nabla\cdot\boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{1.17}$$

The curl of a vector $\boldsymbol{F}$ is defined as

$$\text{Curl } \boldsymbol{F} = \nabla\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \boldsymbol{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \tag{1.18}$$

While discussing the curl of a vector, it is interesting to examine the physical meaning of the curl of the velocity vector of a rotating body, $\boldsymbol{V}$. To do this, $\boldsymbol{V}$ is expressed in terms of rotating velocity $\boldsymbol{\omega}$ and position vector $\boldsymbol{r}$, then

$$\begin{aligned} \boldsymbol{V} &= \boldsymbol{\omega} \times \boldsymbol{r} \\ &= \boldsymbol{i}(\omega_2 z - \omega_3 y) + \boldsymbol{j}(\omega_3 x - \omega_1 z) + \boldsymbol{k}(\omega_1 y - \omega_2 x) \end{aligned}$$

where ω_1, ω_2, and ω_3 are the components of $\boldsymbol{\omega}$, and x, y, and z are the components of $\boldsymbol{r}$. The computation of curl $\boldsymbol{V}$ gives

$$\begin{aligned} \nabla \times \boldsymbol{V} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix} \\ &= \boldsymbol{i}2\omega_1 + \boldsymbol{j}2\omega_2 + \boldsymbol{k}2\omega_3 = 2\boldsymbol{\omega} \end{aligned} \tag{1.19}$$

Therefore $\nabla \times \boldsymbol{V}$ is related with rotational velocity and is known as vorticity in fluid mechanics.

1.3 Statics and Dynamics

Statics is the study of objects at rest or in equilibrium under the actions of forces and/or torques. The equations of statics for different dimensions of space are summarized as follows.

For a one-dimensional problem,

$$\sum F = 0 \tag{1.20}$$

For a two-dimensional problem,

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \sum M_o = 0 \tag{1.21}$$

where F_x, F_y are the components of force in the x and y axes, respectively, and M_o is the moment with respect to a reference axis o perpendicular to the x–y plane.

For a three-dimensional problem,

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \sum F_z = 0 \tag{1.22}$$

$$\sum M_{xx} = 0, \qquad \sum M_{yy} = 0, \qquad \sum M_{zz} = 0 \tag{1.23}$$

where M_{xx}, M_{yy}, and M_{zz} are the moments with respect to the x, y, and z axes, respectively. Therefore, in general, there are six unknowns to be determined by six equations for the three-dimensional problem.

Dynamics is the branch of science that studies the physical phenomena of a body or bodies in motion. Dynamics usually includes kinematics and kinetics. Kinematics concerns only the space–time relationship of a given motion of a body, not the forces that cause the motion. Kinetics concerns finding the motion that a given body or bodies will have under the action of given forces, or finding what forces must be applied to produce a prescribed motion.

1.4 Newton's Laws of Motion

Dynamics is based on Newton's laws of motion, which were written by Sir Isaac Newton in the 17th century; however, before stating his laws we must introduce the concept of a "frame of reference." The position, velocity, and acceleration of a particle in space must be described relative to other points within the space; that is, there must exist a frame of reference in the space. Newton's laws of motion apply only when the frame of reference is either fixed in space or moving with constant velocity. Such a frame of reference is called an inertial frame of reference. An Earth-fixed reference frame usually is acceptable as an inertial reference frame for solving many engineering problems even though the Earth is moving relative to the sun with a speed of 29.8 km/s and a radius of curvature of 1.495×10^8 km. Newton's laws of motion are stated as follows:

First law (law of inertia): A particle remains at rest or at a constant velocity if the resultant force acting on the particle is zero.

Second law (the basic equation of motion): The rate of change of a particle's linear momentum is proportional to the force applied to the particle and occurs in the direction of the force.

Third law (law of action and reaction): For every force a particle exerts on another particle, there exists a reaction force back on the first particle; these two forces are equal in magnitude and opposite in direction.

There are advantages to stating the second law as just shown. For example, a body with changing mass with respect to time can accelerate without any external force applied. To substantiate this statement, the equation of motion is written as

$$\frac{\mathrm{d}m\boldsymbol{V}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} + \boldsymbol{V}\frac{\mathrm{d}m}{\mathrm{d}t} = 0 \qquad (1.24)$$
$$m\boldsymbol{a} = -\dot{m}\boldsymbol{V}$$

This result shows that, if the body is a rocket, the thrust of a rocket is the product of the mass flow rate and its velocity, and the direction of thrust is opposite to the velocity. Because of the way the second law is stated, the equation of motion for a particle with constant mass can be written as

$$\boldsymbol{F} = (1/g_c)m\boldsymbol{a}$$

or

$$w = (1/g_c)mg \qquad (1.25)$$

In the preceding equation, if the unit of mass is pounds of mass and that of the force is pounds of force,

$$g_c = 32.174\frac{\text{lbm}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}$$

However, for the International System of Units (SI) and British Gravitational System (BG) units, g_c is reduced to unity and can be omitted in Eq. (1.25).

1.5 D'Alembert's Principle

In statics, we are familiar with

$$\sum \boldsymbol{F} = 0$$

From this we can solve for the three unknowns in three-dimensional space. In dynamics, the equation of motion for a particle with constant mass is written as

$$\sum \boldsymbol{F} = m\boldsymbol{a} \tag{1.26}$$

where $\sum \boldsymbol{F}$ is the sum of the external forces acting on the particle, m is the particle mass, and $\boldsymbol{a}$ is the acceleration of the particle relative to an inertial reference frame. Now, we rewrite the equation as

$$\sum \boldsymbol{F} - m\boldsymbol{a} = 0 \tag{1.27}$$

and consider the term $-m\boldsymbol{a}$ to represent another force known as an inertia force, then Eq. (1.27) simply states that the vector sum of all forces, external and inertial, is zero. Thus, the dynamics problem has been reduced to a statics problem. This conversion in concept is known as D'Alembert's principle. Similarly, for a body in rotation, the equation of motion is

$$\sum \boldsymbol{T} = I\boldsymbol{\alpha} \tag{1.28}$$

where $\sum \boldsymbol{T}$ is the sum of external torques applying on the body, I is the mass moment of inertia of the body with respect to the rotating axis, and $\boldsymbol{\alpha}$ is the angular acceleration of the body. Equation (1.28) also can be written as

$$\sum \boldsymbol{T} - I\boldsymbol{\alpha} = 0 \tag{1.29}$$

Similar to Eq. (1.27), Eq. (1.29) states that the vector sum of all torques, external and inertial, is zero. Furthermore, the combination of Eqs. (1.27) and (1.29) can be applied to solve problems for a body simultaneously undergoing translation and rotation. In conclusion, this change of concept from dynamics to statics greatly simplifies complicated dynamic problems in mechanics.

1.6 Virtual Work

Consider a system of N particles whose positions are specified by Cartesian coordinates $x_1, x_2, \ldots, x_{3N}$. Suppose that there are $3N$ forces $F_1, F_2, \ldots, F_{3N}$ applied to the particles in the direction of each coordinate. The forces are in static equilibrium. Now imagine that at a given instant the system is given arbitrary and small displacements $\delta x_1, \delta x_2, \ldots, \delta x_{3N}$ in the direction of each coordinate. The work done by the applied forces is

$$\delta w = \sum_{i=1}^{3N} F_i \delta x_i \tag{1.30}$$

δw is known as virtual work and the small displacements δx_i are called virtual displacements. Equation (1.30) can be written in vector notation for the virtual work as

$$\delta w = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i \tag{1.31}$$

where $\boldsymbol{F}_i$ is the force applied to particle i and $\delta \boldsymbol{r}_i$ is the virtual displacement.

Similar to particles in a solid, if particles in space are in static equilibrium, they do not move relative to each other. Total force applied to particle i is the combination of the applied force $\boldsymbol{F}_i$ and the internal force

$$\sum_{j=1}^{N} \boldsymbol{F}_{ij} \qquad (j \neq i)$$

due to other particles. Therefore the equation for the total force is

$$(\boldsymbol{F}_T)_i = \boldsymbol{F}_i + (\boldsymbol{F}_c)_i = 0 \tag{1.32}$$

where

$$(\boldsymbol{F}_c)_i = \sum_{j=1}^{N} \boldsymbol{F}_{ij} \qquad (j \neq i)$$

and $(\boldsymbol{F}_T)_i = 0$ because of equilibrium.

Because the total force is zero, the work done by the total force must be zero, that is, $(\boldsymbol{F}_T)_i \cdot \delta \boldsymbol{r}_i = 0$. The virtual work of all the forces as a result of the virtual displacement $\delta \boldsymbol{r}_i$ is

$$\sum_{i=1}^{N} (\boldsymbol{F}_i + \boldsymbol{F}_{ci}) \cdot \delta \boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i + \sum_{i=1}^{N} \boldsymbol{F}_{ci} \cdot \delta \boldsymbol{r}_i = 0 \tag{1.33}$$

The second term of the preceding equation is further explored as follows:

$$\begin{aligned}
\sum_{i=1}^{N} (\boldsymbol{F}_c)_i \cdot \delta \boldsymbol{r}_i &= \sum_{i,j}^{N} (\boldsymbol{F}_{ij}) \cdot \delta \boldsymbol{r}_i \\
&= \cdots \boldsymbol{F}_{k\ell} \cdot \delta \boldsymbol{r}_k + \boldsymbol{F}_{\ell k} \cdot \delta \boldsymbol{r}_\ell + \cdots \\
&= \cdots \boldsymbol{F}_{k\ell} \cdot \delta \boldsymbol{r}_k - \boldsymbol{F}_{k\ell} \cdot \delta \boldsymbol{r}_\ell + \cdots \\
&= \cdots \boldsymbol{F}_{k\ell} \cdot (\delta \boldsymbol{r}_k - \delta \boldsymbol{r}_\ell) + \cdots \\
&= \cdots \boldsymbol{F}_{k\ell} \cdot \delta (\boldsymbol{r}_k - \boldsymbol{r}_\ell) + \cdots
\end{aligned}$$

in which $i = k, j = \ell$ is considered in the first term and $i = \ell, j = k$ in the second term. The symbol $\delta(\boldsymbol{r}_k - \boldsymbol{r}_\ell)$ is the change of $\boldsymbol{r}_k - \boldsymbol{r}_\ell$ in the solid and can occur only in the direction perpendicular to $\boldsymbol{r}_k - \boldsymbol{r}_\ell$, but $\boldsymbol{F}_{k\ell}$ is along $\boldsymbol{r}_k - \boldsymbol{r}_\ell$, hence the dot product must be zero. Therefore,

$$\sum_{i=1}^{N} (\boldsymbol{F}_c)_i \cdot \delta \boldsymbol{r}_i = 0$$

$$\delta w = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = 0 \tag{1.34}$$

i.e., the virtual work of applied forces is zero. The concept of virtual work will be used for the derivation of Lagrange's equations.

Example 1.1

Using the method of virtual work, determine the relationship between the torque T applied to the crank R and the force F applied to the slider in the mechanism to be shown in Fig. 1.1.

Solution. According to the conditions given in Fig. 1.1, the vector forms of torque, force, and displacements can be written as

$$\boldsymbol{T} = -\boldsymbol{k}T, \qquad \delta\boldsymbol{\theta} = \boldsymbol{k}\delta\theta$$

$$\boldsymbol{F} = -\boldsymbol{i}F, \qquad \delta\boldsymbol{x} = \boldsymbol{i}\delta x$$

In static equilibrium, the total virtual work δw is zero, and its equation is

$$\delta w = -T\delta\theta - F\delta x = 0 \tag{1.35}$$

From the given geometry, we have

$$x = R\cos\theta + L\cos\phi$$

$$R\sin\theta = h = L\sin\phi$$

Solving the two equations, we obtain

$$\cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-(R/L)^2\sin^2\theta}$$

$$x = R\cos\theta + L\sqrt{1-(R/L)^2\sin^2\theta}$$

Differentiating the equation for x, we have

$$\delta x = -R\sin\theta\,\delta\theta - (R^2/L)\frac{\sin\theta\cos\theta}{\sqrt{1-(R/L)^2\sin^2\theta}}\delta\theta \tag{1.36}$$

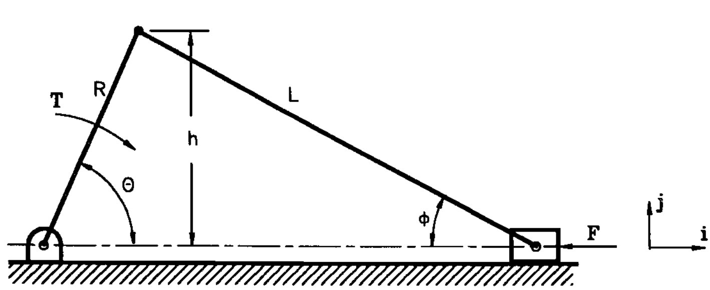

Fig. 1.1 Crank–slider mechanism.

Substituting Eq. (1.36) into Eq. (1.35) and simplifying, we find the required relationship between the torque and the force acting on the slider as

$$T = FR\sin\theta\left\{1 + \frac{R\cos\theta}{L\sqrt{1-(R/L)^2\sin^2\theta}}\right\} \tag{1.37}$$

Problems

1.1. Determine a unit vector perpendicular to the plane passing through (a, 0, 0), (0, b, 0), and (0, 0, c).

1.2. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are defined as follows:

$$\boldsymbol{a} = 2\boldsymbol{i} - 4\boldsymbol{k}, \qquad \boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$$

(a) Find the scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b}$.
(b) Find the angle between the positive directions of the vectors.

1.3. Find the moment of the force $\boldsymbol{F} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, acting at the point (1, 1, 2), about the z axis in arbitrary units.

1.4. Prove that $\boldsymbol{u} \times (\nabla \times \boldsymbol{v}) = \nabla(\boldsymbol{u} \cdot \boldsymbol{v}) - \boldsymbol{u} \cdot \nabla\boldsymbol{v}$, if $\boldsymbol{u}$ is constant.

1.5. Determine a unit vector in the plane of the vectors $\boldsymbol{i} + \boldsymbol{k}$, and $\boldsymbol{j} + \boldsymbol{k}$, and perpendicular to vector $\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$.

1.6. Let $\boldsymbol{r}$ represent the position vector of a moving point mass M, subject to a force $\boldsymbol{F}$. If $\boldsymbol{L}$ denotes the moment of the momentum $m\boldsymbol{v}$ about 0, prove that

$$\frac{d\boldsymbol{L}}{dt} = \frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{M}$$

where $\boldsymbol{M}$ is the moment of the force $\boldsymbol{F}$ about 0.

1.7. Do the following:
(a) Find the unit vector normal to the plane $Ax + By + Cz = D$.
(b) Prove that the shortest distance from the point $P_0(x_0, y_0, z_0)$ to the plane $Ax + By + Cz = D$ is given by

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$$

where the point P_0 is located above the plane. **HINT:** Let $P_1(x_1, y_1, z_1)$ be any point on the plane and determine the distance by letting P_0P_1 along the normal from the plane.

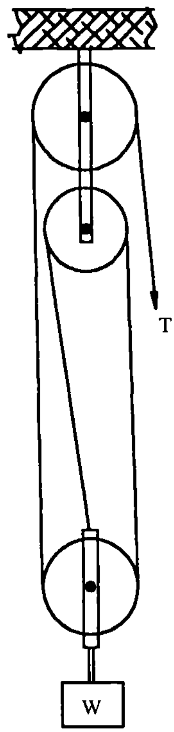

Fig. P1.8

1.8. A light cable passes around a pulley mounted on smooth bearings as shown in Fig. P1.8. The tension on both sides of the pulley is equal. Using the method of virtual work, find the displacement of the cable with tension T in terms of the vertical displacement of weight W. Assume that the pulleys and cable are light and the distance between the upper and lower pulleys is so great that the cables may be regarded as vertical.

1.9. A framework $ABCD$ consists of four equal, light rods smoothly joined together to form a square. It is suspended from a peg at A, and a weight W is attached to C. Further, the framework is kept in shape by a light rod connecting B and D. Determine the force exerted in this rod. **HINT:** The method of virtual work may be applied if the rod BD is removed and external forces are supplied to the joint B and D.

1.10. Consider a U-joint connecting two shafts that are not along a straight line as shown in Fig. P1.10. AB is a shaft, branching into the fork BCD; $A'B'$ is another axis, with fork $B'C'D'$. These forks are connected by a rigid body composed of two bars CD, $C'D'$, joined perpendicularly at their common center O. The lines AB, $A'B'$ meet at O and are perpendicular to CD, $C'D'$, respectively. There are smooth bearings at CD, $C'D'$ and the axes AB, $A'B'$ are free to turn in

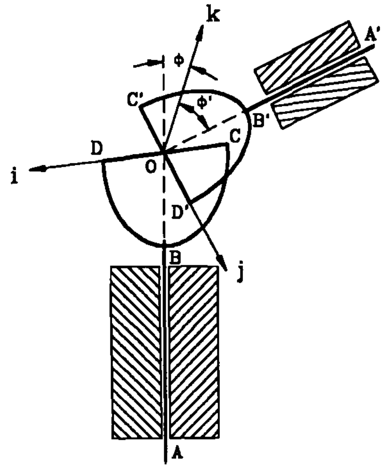

Fig. P1.10

smooth bearings. With the use of the method of virtual work, determine the torque transmitted through the joint. **HINT:** The velocity at the point D must be the same as rotated from two rigid bodies $ABCD$ and $CDC'D'$. Similarly, the velocity at D' must be the same from $A'B'C'D'$ and $CDC'D'$. Establish the virtual angular displacements from two shafts by equating the rotational displacements of CD and $C'D'$.

2
Kinematics and Dynamics of a Particle

A *particle* is defined as a material point without dimensions but containing a definite quantity of matter. Strictly speaking, a particle cannot exist, because a definite amount of matter must occupy some space. When the size of a body is extremely small compared with its range of motion, however, it may be considered as a particle in certain cases. For example, although stars and planets are many thousands of miles in diameter, they are so small compared with their range of motion that they are often considered as particles in space.

This chapter covers material that should not be totally new to the reader. Coverage in some areas, such as kinematics of a particle in cylindrical and spherical coordinates, is more in depth than that given in an introductory course in dynamics. The relationship between curvilinear and rectangular coordinates for unit vectors is introduced in Section 2.1 so that velocities and accelerations in curvilinear coordinates are obtained easily. Some relatively modern examples illustrating particle dynamics are given in Section 2.2 although we expect the reader to have some familiarity with particle dynamics from studying elementary dynamics. Examples concerning missiles and space vehicles given here will be revisited in examples describing midair collisions of missiles in the next chapter. The change of angular momentum caused by applied moment is discussed in Section 2.3. Example 2.3 shows that the side force existing between a sliding block and rotating rod can be very significant. Work and conservative force are reviewed in Sections 2.4 and 2.5. They are useful for understanding the concept of potential energy used in Lagrangian equations.

2.1 Kinematics of a Particle

The location of a particle in three-dimensional space always can be specified by a position vector $\boldsymbol{r}$. Its velocity $\boldsymbol{v}$ is defined as

$$\boldsymbol{v} \equiv \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \tag{2.1}$$

Similarly the acceleration of the particle is defined as

$$\boldsymbol{a} \equiv \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \tag{2.2}$$

Now, let us develop expressions for velocity and acceleration of a particle in different coordinate systems.

Cartesian Coordinates

The position vector of a particle is

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \tag{2.3}$$

Note that $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ are constant vectors. The velocity $\boldsymbol{v}$ is therefore

$$\boldsymbol{v} = \boldsymbol{i}\frac{\mathrm{d}x}{\mathrm{d}t} + \boldsymbol{j}\frac{\mathrm{d}y}{\mathrm{d}t} + \boldsymbol{k}\frac{\mathrm{d}z}{\mathrm{d}t} = \boldsymbol{i}\dot{x} + \boldsymbol{j}\dot{y} + \boldsymbol{k}\dot{z} \tag{2.4}$$

and the acceleration is

$$\boldsymbol{a} = \boldsymbol{i}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \boldsymbol{j}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \boldsymbol{k}\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = \boldsymbol{i}\ddot{x} + \boldsymbol{j}\ddot{y} + \boldsymbol{k}\ddot{z} \tag{2.5}$$

Cylindrical Coordinates

The position vector of a particle in cylindrical coordinates is

$$\boldsymbol{r} = \rho\boldsymbol{e}_\rho + z\boldsymbol{k} \tag{2.6}$$

where ρ is the projected length of $\boldsymbol{r}$ in the x–y plane, as shown in Fig. 2.1.

The unit vector is $\boldsymbol{e}_\rho$ along ρ in the x–y plane and can be expressed in terms of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ as

$$\boldsymbol{e}_\rho = \cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j} \tag{2.7}$$

A unit vector that is perpendicular to $\boldsymbol{e}_\rho$ but lies in the x–y plane is denoted by $\boldsymbol{e}_\phi$ as shown. It also can be expressed in terms of $\boldsymbol{i}, \boldsymbol{j}$ as

$$\boldsymbol{e}_\phi = -\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j} \tag{2.8}$$

The velocity of a particle in cylindrical coordinates is

$$\begin{aligned}\boldsymbol{v} &= \dot{\rho}\boldsymbol{e}_\rho + \rho\dot{\boldsymbol{e}}_\rho + \dot{z}\boldsymbol{k} \\ &= \dot{\rho}[\cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j}] + \rho\dot{\phi}[-\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j}] + \dot{z}\boldsymbol{k} \\ &= \dot{\rho}\boldsymbol{e}_\rho + \rho\dot{\phi}\boldsymbol{e}_\phi + \dot{z}\boldsymbol{k}\end{aligned} \tag{2.9}$$

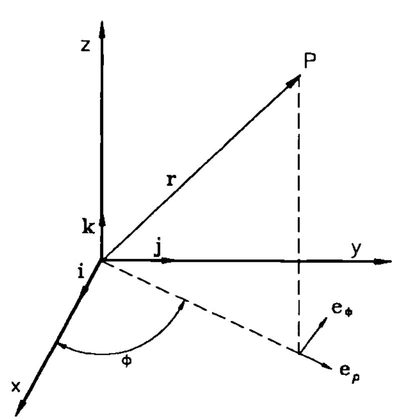

Fig. 2.1 Cylindrical coordinates.

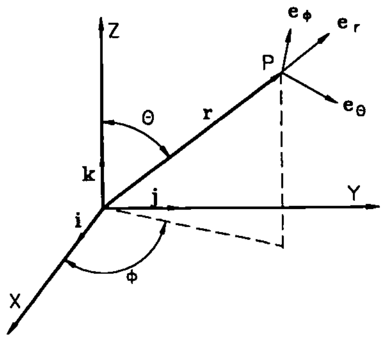

Fig. 2.2 Spherical coordinates.

Its acceleration in cylindrical coordinates is then

$$a \equiv \frac{dv}{dt} = (\ddot{\rho} - \rho\dot{\phi}^2)e_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})e_\phi + \ddot{z}k \tag{2.10}$$

Spherical Coordinates

The unit vectors in spherical coordinates are denoted by e_r, e_θ, and e_ϕ. The e_r is in the direction of position vector r; hence

$$r = re_r \tag{2.11}$$

The e_θ is in the plane containing r and the z axis, but is perpendicular to e_r, as shown in Fig. 2.2. The e_ϕ is perpendicular to both e_r and e_θ. Therefore, they also can be expressed in terms of unit constant vectors i, j, and k as

$$e_r = \sin\theta\, \cos\phi i + \sin\theta\, \sin\phi j + \cos\theta k \tag{2.12}$$

$$e_\theta = \cos\theta\, \cos\phi i + \cos\theta\, \sin\phi j - \sin\theta k \tag{2.13}$$

$$e_\phi = -\sin\phi i + \cos\phi j \tag{2.14}$$

During the differentiating of r with respect to time, Eqs. (2.12–2.14) are used. With some details omitted, the velocity of a particle in spherical coordinates is found to be

$$\begin{aligned} v &= \dot{r}e_r + r\dot{e}_r \\ &= \dot{r}e_r + r(\dot{\theta}e_\theta + \dot{\phi}\sin\theta e_\phi) \\ &= \dot{r}e_r + r\dot{\theta}e_\theta + r\dot{\phi}\sin\theta e_\phi \end{aligned} \tag{2.15}$$

Similarly, the acceleration of a particle can be obtained through the differentiation of v with respect to time and can be expressed as

$$\begin{aligned} a \equiv\ & e_r(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta) \\ & + e_\theta(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2 \sin\theta\, \cos\theta) \\ & + e_\phi(2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\, \cos\theta + r\, \sin\theta\, \ddot{\phi}) \end{aligned} \tag{2.16}$$

Note that with the use of Eqs. (2.12–2.14), Eq. (2.16) can be reduced to Eq. (2.5).

2.2 Particle Kinetics

In general, a force $\boldsymbol{F}$ acting on a point mass m is a function of position, velocity, and time. The equation of motion for the particle with constant mass can be written simply as

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \tag{2.17}$$

Many cases are studied in introductory dynamics. Let us study a few special cases in the following examples.

Example 2.1

Consider a missile moving in space as a particle with a mass decreasing constantly. The thrust applied is constant in magnitude and always in the direction of the particle's velocity. The coordinates are chosen such that the x–z plane contains the trajectory with the z axis perpendicular to the ground. Find the trajectories of the missile for thrust F = 14,500, 15,000, and 15,500 N, respectively. The initial conditions of the missile are $m_0 = 1000$ kg and $v_0 = 150$ m/s at an angle of 80 deg with the x axis. The mass decreasing rate of $\dot{m} = 3$ kg/s.

Solution. The equation of motion for the missile is

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = F\frac{\boldsymbol{v}}{|\boldsymbol{v}|} - mg\boldsymbol{k} \tag{2.18}$$

or

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = F\frac{v_x}{\sqrt{v_x^2 + v_z^2}} \tag{2.19}$$

$$m\frac{\mathrm{d}v_z}{\mathrm{d}t} = F\frac{v_z}{\sqrt{v_x^2 + v_z^2}} - mg \tag{2.20}$$

$$m = m_0 - \dot{m}t \tag{2.21}$$

Equations (2.19) and (2.20) are nonlinear and cannot be solved analytically. However, they can be integrated numerically by the Runge–Kutta method given in the Appendix A. The trajectory then can be obtained as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x \tag{2.22}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v_z \tag{2.23}$$

integrated together with Eqs. (2.19) and (2.21). Three trajectories are obtained for the three different values of thrust. The results are given in Fig. 2.3. In the numerical integration the increment of time used is 0.01 s and the total duration is more than 160 s. A convergence check is performed before the results are calculated.

Example 2.2

Suppose that a space vehicle is moving from outer orbit into the atmosphere. The aerodynamic drag acting on the vehicle is proportional to the velocity squared.

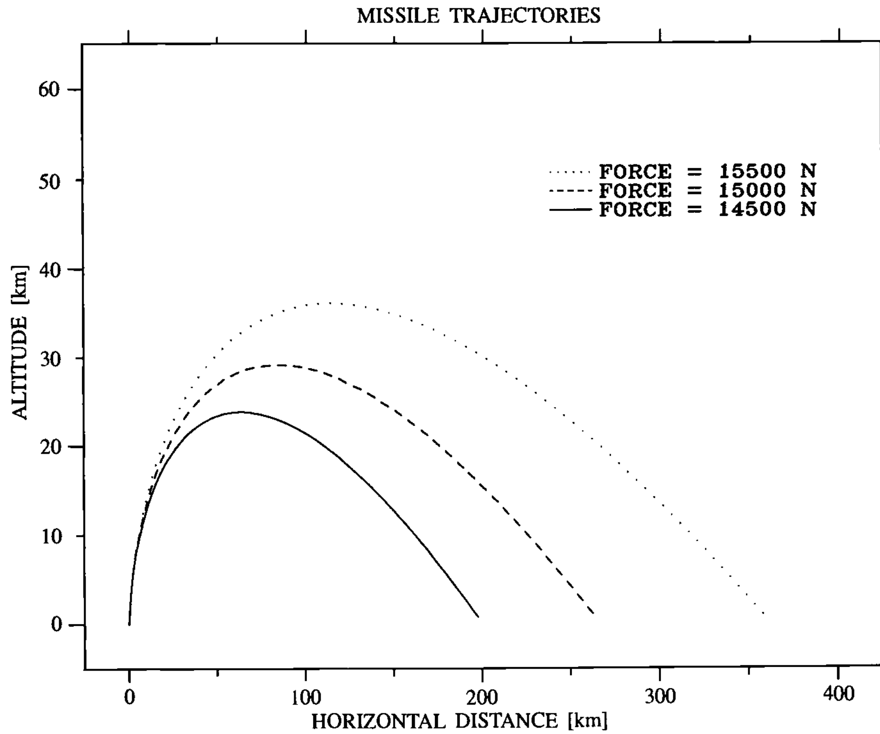

Fig. 2.3 Trajectories of the missile.

The coordinates are chosen such that the x–y plane contains the trajectory and the y axis is along $-\boldsymbol{g}$ as shown in Fig. 2.4. Determine the trajectories of the space vehicle as it descends with initial velocities of 7000, 8000 and 9000 m/s. The initial location of the vehicle is $x_0 = 0$, $y_0 = 20$ km. And its initial trajectory is always parallel to the ground.

Solution. According to the given conditions, the equations of motion can be written as

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg\sin\alpha - H(v) \tag{2.24}$$

and

$$v^2/R = g\cos\alpha \tag{2.25}$$

where R is the radius of curvature of the trajectory and $H(v)$ is the aerodynamic drag of the vehicle:

$$H(v) = mkv^2 \tag{2.26}$$

where k, which should be a function of altitude, is considered as a constant for this example. Equations (2.24) and (2.26) lead us to

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g \sin\alpha - kv^2 \tag{2.27}$$

The value of k is estimated to be $1.5 \times 10^{-6}(\mathrm{m}^{-1})$. However,

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}, \qquad \frac{v^2}{R} = \frac{(\mathrm{d}s/\mathrm{d}t)^2}{(\mathrm{d}s/\mathrm{d}\alpha)} = \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\alpha}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\alpha}{\mathrm{d}t} = v\frac{\mathrm{d}\alpha}{\mathrm{d}t}$$

where $\mathrm{d}s$ is the infinitesimal displacement along the trajectory and α is the angle between the velocity and the horizontal line as shown in Fig. 2.4. Substituting the preceding equation into Eq. (2.25), we obtain

$$v\frac{\mathrm{d}\alpha}{\mathrm{d}t} = g\cos\alpha$$

or

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{g\cos\alpha}{v} \tag{2.28}$$

Note that we also have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x = v\cos\alpha \tag{2.29}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v_y = -v\sin\alpha \tag{2.30}$$

Equations (2.27–2.30) can be integrated by the Runge–Kutta method given in Appendix A to find $x(t)$, $y(t)$, which is the trajectory of the vehicle with time t as the parameter. The result of numerical integration is given in Fig. 2.5.

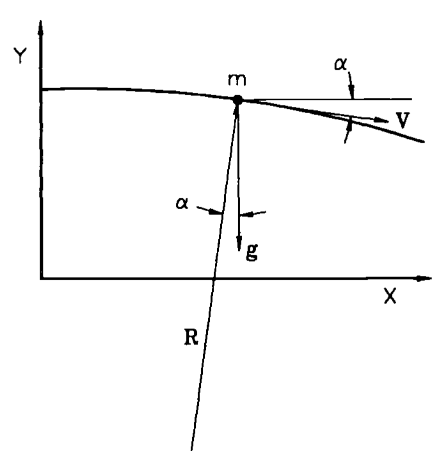

Fig. 2.4 Coordinates of the space vehicle.

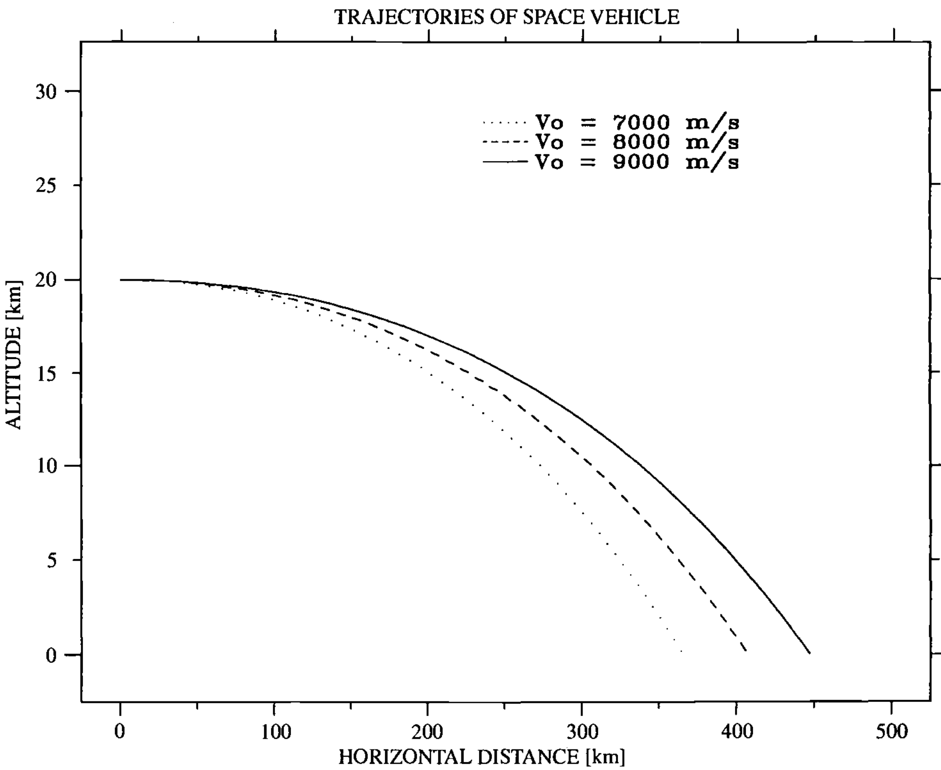

Fig. 2.5 Trajectory of the space vehicle.

2.3 Angular Momentum (Moment of Momentum) of a Particle

Another aspect of the particle dynamics is the change of angular momentum with respect to a certain axis when an external moment is applied. The angular momentum or moment of momentum of a particle is defined as

$$\boldsymbol{H} = \boldsymbol{r} \times m\boldsymbol{v} \tag{2.31}$$

where $\boldsymbol{r}$ is the position vector from the axis to the particle. The relationship between angular and linear momentum is shown in Fig. 2.6. The moment produced by the force applied to the particle is

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$$

where $\boldsymbol{F} = m(\mathrm{d}\boldsymbol{v}/\mathrm{d}t)$. Differentiating Eq. (2.31) leads to

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times m\boldsymbol{v} + \boldsymbol{r} \times m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{M}$$

The term $(\mathrm{d}\boldsymbol{r}/\mathrm{d}t) \times m\boldsymbol{v}$ is dropped because $(\mathrm{d}\boldsymbol{r}/\mathrm{d}t) = \boldsymbol{v}$ and $\boldsymbol{v} \times \boldsymbol{v} = 0$. Hence

$$\boldsymbol{M} = \frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} \tag{2.32}$$

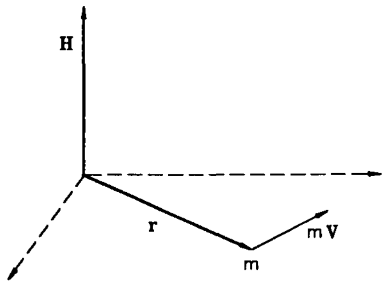

Fig. 2.6 Relationship between angular and linear momentums.

Example 2.3

To illustrate the meaning of Eq. (2.32), let us consider a block as a particle sliding on a straight rod without friction at a uniform velocity of 30 ft/s, as shown in Fig. 2.7. The rod is in the x–y plane, which is perpendicular to the gravitational force. The angular velocity of the rod is 50 rad/s. The position of the block is 6 in. away from the rotating axis. Determine the force between the block and the rod if the mass of the block is 1/30 slug.

Solution. Rewrite Eq. (2.31) as

$$\boldsymbol{H} = \boldsymbol{r} \times m\boldsymbol{v}$$

For this example, it is convenient to use cylindrical coordinates. The position vector of the particle at time t is $\boldsymbol{r} = r\boldsymbol{e}_\rho$. Its velocity is

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_\rho + r\omega\boldsymbol{e}_\phi$$

Hence

$$\boldsymbol{H} = r\boldsymbol{e}_\rho \times m(\dot{r}\boldsymbol{e}_\rho + r\omega\boldsymbol{e}_\phi) = mr^2\omega\boldsymbol{k}$$

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = rF\boldsymbol{k} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H} = 2mr\dot{r}\omega\boldsymbol{k}$$

$$F = 2m\dot{r}\omega = 2(1/30)(30)(50) = 100 \qquad \text{(lbf)}$$

Therefore, the force between the block and the rod is 100 lbf.

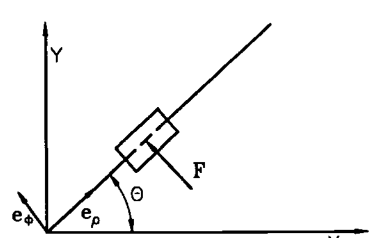

Fig. 2.7 Block sliding on a rotating rod.

2.4 Work and Kinetic Energy

Work usually is defined as a force F acting through a displacement x with the displacement occurring in the direction of the force. That is,

$$W = \int_1^2 F \cdot \mathrm{d}x$$

Using vector notation, the equivalent expression is

$$W = \int_1^2 \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

In general, if $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$ are not in the same direction, only the component of $\mathrm{d}\boldsymbol{r}$ along $\boldsymbol{F}$ will contribute to the work. If the force is applied to a particle with a constant mass, then

$$\boldsymbol{F} = m\boldsymbol{a} = m\dot{\boldsymbol{v}}$$

and the work done by the force is

$$\begin{aligned} W &= \int_1^2 m\dot{\boldsymbol{v}} \cdot \mathrm{d}\boldsymbol{r} = \int_1^2 m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{r} \\ &= \int_1^2 m\mathrm{d}\boldsymbol{v} \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \int_1^2 m\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{v} \\ &= \tfrac{1}{2}m\left(v_2^2 - v_1^2\right) = T_2 - T_1 \end{aligned} \tag{2.33}$$

where T is the kinetic energy of the particle. Equation (2.33) says that the change in kinetic energy of a particle moving from one point to another is equal to the work done by the force acting on the particle.

2.5 Conservative Forces

Suppose that a particle m moves from A to B as shown in Fig. 2.8 and a force $\boldsymbol{F}$ is applied to the particle during the process. Then the work is

$$W = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

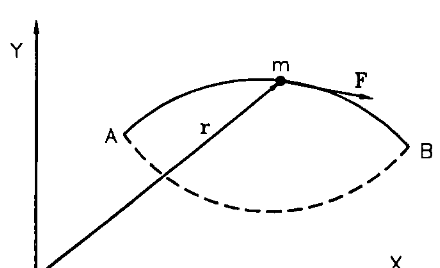

Fig. 2.8 Moving paths of a particle.

which is the line integral from A to B and may be represented by the solid line AB in Fig. 2.8.

On the other hand, if

$$\int_B^A \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = -\int_A^B \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$$

where the line integral from B to A may be represented by the dotted line BA in the figure, then

$$\oint \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = 0$$

This means that the line integral of $\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$ over a closed path is zero. According to Stoke's theorem given in Appendix B,

$$\oint \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \iint_s \nabla \times \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s} \tag{2.34}$$

where s is the area bounded by the closed path in the line integral. If the closed path is arbitrarily chosen, then

$$\nabla \times \boldsymbol{F} = 0$$

is true everywhere. According to vector analysis, the force $\boldsymbol{F}$ must be a gradient of a scalar function, i.e.,

$$\boldsymbol{F} = \nabla\phi$$

where ϕ is a scalar function to be identified. Force with this property is called a conservative force. Work done by such a force is

$$\begin{aligned} w &= \int_A^B \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_A^B \nabla\phi\cdot \mathrm{d}\boldsymbol{r} \\ &= \int_A^B \left(\frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z\right) = \int_A^B \mathrm{d}\phi = \phi_B - \phi_A \end{aligned} \tag{2.35}$$

Combining the preceding equation with Eq. (2.33) gives

$$\phi_B - \phi_A = T_B - T_A$$

or

$$T_A - \phi_A = T_B - \phi_B \tag{2.36}$$

To identify ϕ, let us recall the principle of conservation of mechanical energy, which states that the sum of kinetic and potential energies is constant for a conservative system. Put in equational form,

$$T_A + V_A = T_B + V_B \tag{2.37}$$

where V is the potential energy of the particle. Comparing Eqs. (2.36) and (2.37), we find

$$\phi = -V$$

Therefore $\boldsymbol{F} = -\nabla V$. A conservative force is equal to a gradient of potential energy with a change of sign.

Problems

2.1. Prove that the velocity expressed in cylindrical coordinates

$$\boldsymbol{v} = \dot{\rho}\boldsymbol{e}_\rho + \rho\dot{\phi}\boldsymbol{e}_\phi + \dot{z}\boldsymbol{k}$$

can be converted to the expression of velocity in Cartesian coordinates.

2.2. Prove that the expression of acceleration in spherical coordinates, Eq. (2.16), can be converted to

$$\boldsymbol{a} = \boldsymbol{i}\ddot{x} + \boldsymbol{j}\ddot{y} + \boldsymbol{k}\ddot{z}$$

2.3. The position vector of a moving particle is

$$\boldsymbol{r} = \boldsymbol{i}a \cos \omega t + \boldsymbol{j}b \sin \omega t$$

where a, b, and ω are constants.

(a) Find the velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ and prove that $\boldsymbol{r} \times \boldsymbol{v}$ is constant

(b) Show that the acceleration is directed toward the origin and is proportional to the distance from the origin.

2.4. At a certain instant, a particle of mass m moving freely in a vertical plane under a constant gravity is at a height h above the ground and has a speed v. Use the principle of energy to find its speed when it strikes the ground.

2.5. Two masses, m_1 and m_2, are connected by a massless, inextensible rope that passes over a pulley, as shown in Fig. P2.5. Neglecting the mass and the bearing friction of the pulley, find the acceleration of m_1 as the system moves under the action of gravity.

2.6. A constant force is applied to a point mass so that the mass is accelerating. Two frames of reference are chosen for consideration. One is a fixed reference frame; the x axis is oriented along the acceleration. The other is moving with a constant velocity along the negative x direction of the fixed reference frame. However, they coincide at the beginning of observation.

(a) Find the velocity and position of the particle as a function of time in both reference frames.

(b) Find the work done by the force during a time interval t in both frames.

(c) Are the results of (b) different in the two frames? If so, are the laws of mechanics different in the two inertial frames of reference? Explain your answer.

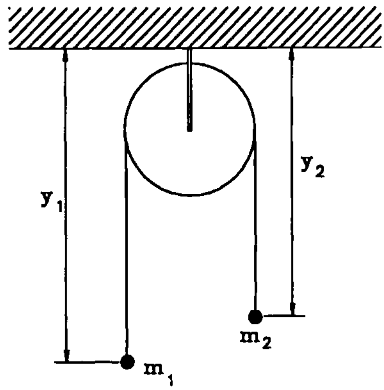

Fig. P2.5

2.7. Suppose that a missile is launched with the initial conditions: constant thrust, constant mass flow rate at the nozzle exit, and a proper launch angle. What will be the force exerting on the missile after the propellant is burned. Formulate the equations for describing the trajectory of the missile.

2.8. Find the best launch angle for a missile to reach the maximum horizontal distance through numerical integration. The fourth-order Runge–Kutta method is to be used for integration. The initial conditions are $F = 15{,}000\,\text{N}$, $M_0 = 1000\,\text{kg}$, $V_0 = 150$ m/s, and $\dot{m} = 3$ kg/s. At the time of burnout, the mass of missile is $M_f = 300$ kg. Plot the trajectory of the missile at the best launch angle.

2.9. Do the following:

(a) Using Green's theorem, prove that

$$\frac{1}{2}\oint_c (x\,\mathrm{d}y - y\,\mathrm{d}x) = A$$

where A is the area enclosed by the curve c.

(b) Find the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

2.10. Show that

(a)

$$\frac{1}{3}\oint_c (xy\,\mathrm{d}y - y^2\mathrm{d}x) = A\bar{y}$$

(b)

$$\frac{1}{4}\oint_c (xy^2\mathrm{d}y - y^3\mathrm{d}x) = I_x$$

where A is the area bounded by C, $(\bar{x}, \bar{y})$ is its centroid, and I_x its moment of inertia about x axis.

2.11. If $\boldsymbol{e}_r$, $\boldsymbol{e}_\phi$, and $\boldsymbol{e}_\theta$ are the unit vectors in spherical coordinates, show that the unit vectors in Cartesian coordinates can be written as

$$\boldsymbol{i} = (\boldsymbol{e}_r \sin\theta + \boldsymbol{e}_\theta \cos\theta)\cos\phi - \boldsymbol{e}_\phi \sin\phi$$

$$\boldsymbol{j} = (\boldsymbol{e}_r \sin\theta + \boldsymbol{e}_\theta \cos\theta)\sin\phi + \boldsymbol{e}_\phi \cos\phi$$

$$\boldsymbol{k} = \boldsymbol{e}_r \cos\theta - \boldsymbol{e}_\theta \sin\theta$$

2.12. A particle of mass moves in a plane under the action of a force with components

$$F_x = -K^2(2x + y), \qquad F_y = -K^2(x + 2y)$$

where K is a constant. Consider that the force is conservative. What is the potential energy?

3
Dynamics of a System of Particles

IN this chapter we shall study the motion of a system of n particles subjected to external and internal forces. These internal forces, which arise from the interaction between the particles, obey Newton's third law of motion. Therefore, when all of the particles are considered as a unit, the internal forces add up to zero. Next, we shall discuss the angular momentum of a system of n particles. This subject plays an important role in studying the rotational motion of a solid body later in this book.

The collision of missiles in midair is analyzed in Section 3.2. The example illustrates that as two missile sites are a few hundred kilometers apart, the spherical surface of the Earth must be considered in the determination of the launching angle. Otherwise the second missile will not collide with the first missile if the launching angle is set according to the flat ground formulation. The gravitational force studied in the missile-to-missile collision is approximated to be always parallel to the z axis. The gravitational force, however, is easily modeled toward the center of Earth with a major component in the $\boldsymbol{k}$ direction and a small component in $\boldsymbol{i}$ direction where $\boldsymbol{i}$ and $\boldsymbol{k}$ are along the Cartesian coordinates chosen at the missile site. To simplify calculation, each missile is modeled as a particle so that the effects of air drag and the thrust of side jets on the missile can be neglected. The thrust is treated as a constant in the section. Precise treatment of the gravitation force in this case is unnecessary. The computer program used to solve this example, however, is easily modified to handle forces in precise forms. In the study of missile collision, two missiles must be addressed in the same coordinate system. Based on the knowledge of vector algebra, the conversion of coordinates is formulated and discussed in Section 3.1.

In the presence of two particles, there exist gravitational force and potential between them. We shall discuss these concepts in Section 3.4. It is interesting to mention that the gravitational force outside a solid sphere, such as Earth, is equivalent to that of a point mass with the same mass occurring at the center of the solid sphere; on the other hand the gravitational force is zero for a point mass located at the center of the solid sphere.

The collisions of solid spheres are discussed in Section 3.5. Both elastic and inelastic collisions are considered. Special emphasis is placed on automobile collision, which is closely related to our daily life.

3.1 Conversion of Coordinates

Before studying the collision of two missiles in the next section, we need to discuss the conversion of coordinates. Because two missile sites are a few hundred kilometers apart, each missile may be described by its own coordinate system first; then they must be converted into one set of coordinates. The procedure of establishing the relationship between the two sets of coordinates is referred to as the conversion of coordinates.

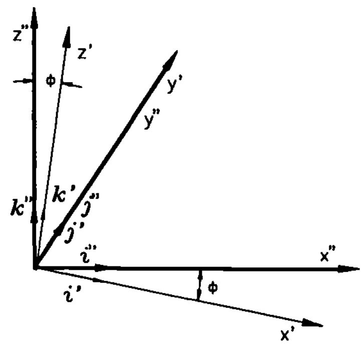

Fig. 3.1a $x''y''z''$ **rotated with respect to** j'' **by** ϕ.

Consider that the coordinate system XYZ is to exist permanently and the coordinate system xyz is to be converted. Starting from a general case, a system $x''y''z''$ is parallel to XYZ, i.e., $\boldsymbol{i}''//\,\boldsymbol{i}$, $\boldsymbol{j}''//\,\boldsymbol{j}$, $\boldsymbol{k}''//\,\boldsymbol{k}$. First, $x''y''z''$ is rotated with respect to the $\boldsymbol{j}''$ axis by an angle of ϕ as shown in Fig. 3.1a. Then, the new coordinates $x'y'z'$ are rotated with respect to the $\boldsymbol{k}'$ axis by an angle of $\boldsymbol{\theta}$. After this rotation, the final coordinates are denoted by xyz as shown in Fig. 3.1b.

The relationship between XYZ and xyz is shown in Fig. 3.2. The position vector $\boldsymbol{R}$ locates the origin of xyz in XYZ. The position of a point P in xyz is denoted by the position vector ρ as

$$\rho = \boldsymbol{i}_\rho x + \boldsymbol{j}_\rho y + \boldsymbol{k}_\rho z$$

In terms of XYZ, the position vector of point P is $\boldsymbol{r}$ and we have

$$\boldsymbol{r} = \boldsymbol{R} + \rho \tag{3.1}$$

Writing in terms of their components, Eq. (3.1) becomes

$$X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k} = X_0\boldsymbol{i} + Y_0\boldsymbol{j} + Z_0\boldsymbol{k} + x\boldsymbol{i}_\rho + y\boldsymbol{j}_\rho + z\boldsymbol{k}_\rho \tag{3.2}$$

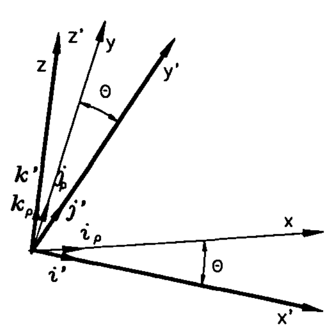

Fig. 3.1b $x'y'z'$ **rotated with respect to** $\boldsymbol{k}'$ **by** θ.

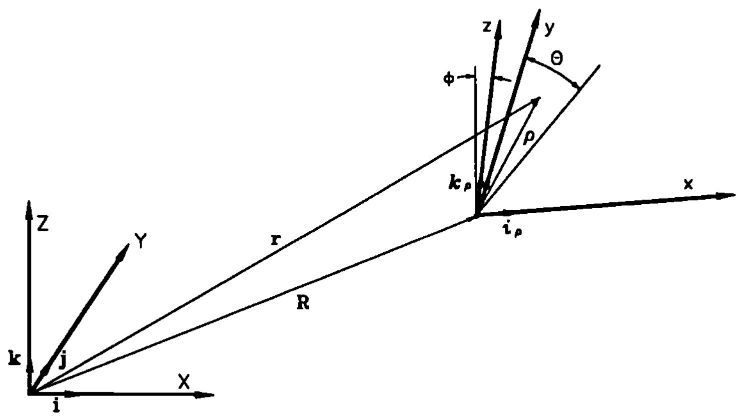

Fig. 3.2 Relationship between *XYZ* and *xyz* systems.

Note that in the preceding equation,

$$\begin{aligned}
\boldsymbol{i}_\rho &= \cos\theta \boldsymbol{i}' + \sin\theta \boldsymbol{j}' \\
&= \cos\theta(\cos\phi \boldsymbol{i} - \sin\phi \boldsymbol{k}) + \sin\theta \boldsymbol{j} \\
&= \cos\theta\cos\phi \boldsymbol{i} + \sin\theta \boldsymbol{j} - \cos\theta\sin\phi \boldsymbol{k} \\
\boldsymbol{j}_\rho &= \cos\theta \boldsymbol{j}' - \sin\theta \boldsymbol{i}' \\
&= -\sin\theta\cos\phi \boldsymbol{i} + \cos\theta \boldsymbol{j} + \sin\theta\sin\phi \boldsymbol{k} \\
\boldsymbol{k}_\rho &= \sin\phi \boldsymbol{i} + \cos\phi \boldsymbol{k}
\end{aligned}$$

In simplifying the preceding equations, we have used the relations $\boldsymbol{i}'' = \boldsymbol{i}$, $\boldsymbol{j}'' = \boldsymbol{j}$, $\boldsymbol{k}'' = \boldsymbol{k}$.

To obtain the X, Y, Z components of $\boldsymbol{r}$, we take the scalar product of the unit vector with Eq. (3.2) as the following:

The scalar product of $\boldsymbol{i}$ with Eq. (3.2) gives

$$\begin{aligned}
X &= X_0 + x\cos(\boldsymbol{i}_\rho, \boldsymbol{i}) + y\cos(\boldsymbol{j}_\rho, \boldsymbol{i}) + z\cos(\boldsymbol{k}_\rho, \boldsymbol{i}) \\
&= X_0 + x\cos\theta\cos\phi - y\sin\theta\cos\phi + z\sin\phi
\end{aligned} \tag{3.3}$$

The scalar product of $\boldsymbol{j}$ with Eq. (3.2) gives

$$\begin{aligned}
Y &= Y_0 + x\cos(\boldsymbol{i}_\rho, \boldsymbol{j}) + y\cos(\boldsymbol{j}_\rho, \boldsymbol{j}) + z\cos(\boldsymbol{k}_\rho, \boldsymbol{j}) \\
&= Y_0 + x\sin\theta + y\cos\theta
\end{aligned} \tag{3.4}$$

Finally the scalar product of $\boldsymbol{k}$ with Eq. (3.2) gives

$$\begin{aligned}
Z &= Z_0 + x\cos(\boldsymbol{i}_\rho, \boldsymbol{k}) + y\cos(\boldsymbol{j}_\rho, \boldsymbol{k}) + z\cos(\boldsymbol{k}_\rho, \boldsymbol{k}) \\
&= Z_0 - x\cos\theta\sin\phi + y\sin\theta\sin\phi + z\cos\phi
\end{aligned} \tag{3.5}$$

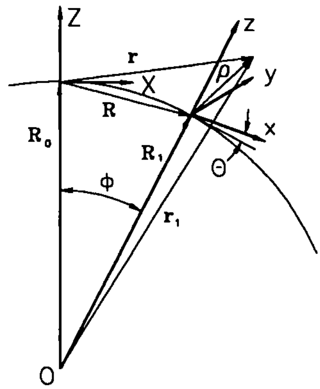

Fig. 3.3 Transfer of coordinates on spherical surface.

In a special case, if there is no rotation with respect to the y axis, i.e., $\phi = 0$, Eqs. (3.3–3.5) reduce to

$$X = X_0 + x\cos\theta - y\sin\theta \tag{3.6}$$

$$Y = Y_0 + x\sin\theta + y\cos\theta \tag{3.7}$$

$$Z = Z_0 + z \tag{3.8}$$

On the other hand, when two coordinate systems are apart by an order of a few hundred kilometers on the surface of the Earth, the effect of the spherical surface must be taken into consideration. Consider that the coordinate systems are on the spherical surface of the Earth as shown in Fig. 3.3. The XYZ system is so chosen that the plane containing x and z axes is the same plane containing R_0, R_1, and R. The unit vector $\boldsymbol{k}$ is along the vector $\boldsymbol{R}_0$ that is pointing from the center of Earth radially to the origin of XYZ. $\boldsymbol{R}_1$ is the position vector of the origin of xyz. Hence

$$\boldsymbol{R}_0 = \boldsymbol{k}R_0$$

$$\boldsymbol{R}_1 = (\boldsymbol{i}\sin\phi + \boldsymbol{k}\cos\phi)R_1$$

$$\begin{aligned} \boldsymbol{R} &= \boldsymbol{R}_1 - \boldsymbol{R}_0 \\ &= \boldsymbol{i}R_1\sin\phi - \boldsymbol{k}(R_0 - R_1\cos\phi) \\ &= \boldsymbol{i}R_0\sin\phi - \boldsymbol{k}R_0(1-\cos\phi) \end{aligned} \tag{3.9}$$

In the preceding equation, it is assumed that the Earth is a perfect sphere, so $\boldsymbol{R}_1$ and $\boldsymbol{R}_0$ are equal. Applying Eqs. (3.3–3.5) with $\boldsymbol{R}$ given in Eq. (3.9), we have the scalar components of $\boldsymbol{r}$ as

$$X = R_0\sin\phi + x\cos\theta\cos\phi - y\sin\theta\cos\phi + z\sin\phi \tag{3.10}$$

$$Y = x\sin\theta + y\cos\theta \tag{3.11}$$

$$Z = -R_0(1-\cos\phi) - x\cos\theta\sin\phi + y\sin\theta\sin\phi + z\cos\phi \tag{3.12}$$

where R_0 is the average radius of Earth and its value is 6371.23 km.

3.2 Collision of Particles in Midair

Study of the collision of two missiles in midair is based on the motions of individual missiles. To simplify the problem let us model them as particles as in the example given in Section 2.2. Although it is known that the second missile is equipped with side jets for adjusting its course, these side thrusts are omitted here. The forces applied on each missile could be very complicated because of variable thrust and air drag. In addition, the mass of a missile is decreasing continuously. However, the model can be simplified greatly by considering that the force applied is constant and the mass ejected from the propulsion system is also at a constant rate. This is an approximate model. Let us study the collision of two missiles with the following example.

Example 3.1

Suppose that a missile is launched from the enemy side, which is designated as the first missile. Through the detection by a satellite, the trajectory can be simulated as given in Example 2.2 with the net thrust of $F = 14{,}500$ N. The coordinates are transferred. Because of the action taken for the determination of the trajectory of the first missile, the time for launching the second missile is delayed by 60 s. To simplify the calculation, the trajectories of the two missiles are assumed to be contained in the same plane, but the launching sites are 200 km apart. The data for the second missile are given as follows: initial mass $m_0 = 1000$ kg, thrust $F = 16{,}000$ N, initial velocity = 300 m/s, and the mass decreasing rate = 3 kg/s. The problem is to determine the launching angle of the second missile so that the two missiles are to collide high above the ground. The conversion of coordinates is treated in two different ways: 1) flat ground and 2) spherical ground.

Solution. 1) Consider that the two launching sites are on flat ground. Each missile is governed by the following equations:

$$m_i \frac{\mathrm{d}V_{xi}}{\mathrm{d}t} = F \frac{V_{xi}}{\sqrt{V_{xi}^2 + V_{zi}^2}} \qquad (i = 1, 2) \tag{3.13}$$

$$m_i \frac{\mathrm{d}V_{zi}}{\mathrm{d}t} = F \frac{V_{zi}}{\sqrt{V_{xi}^2 + V_{zi}^2}} - m_i g \qquad (i = 1, 2) \tag{3.14}$$

$$m_i = m_{i0} - \dot{m}_i t \qquad (i = 1, 2) \tag{3.15}$$

Equations (3.13) and (3.14) are nonlinear and are solved by numerical integration with

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = V_{xi}, \qquad \frac{\mathrm{d}z_i}{\mathrm{d}t} = V_{zi} \tag{3.16}$$

The conditions used for the first missile are

$$(m_1)_0 = 1000 \text{ kg}$$
$$\dot{m}_1 = 3 \text{ kg/s}$$
$$(V_1)_0 = 150 \text{ m/s}$$
$$\alpha_1 = 80 \text{ deg}$$
$$F_1 = 14{,}500 \text{ N}$$

where α is the launching angle measured from x axis. The coordinates are transferred simply by

$$X_1 = X_0 - x_1 \tag{3.17}$$

$$Z_1 = z_1 \tag{3.18}$$

The conditions used for the second missile are

$$(m_2)_0 = 1000 \text{ kg}$$
$$\dot{m}_2 = 3 \text{ kg/s}$$
$$(V_2)_0 = 300 \text{ m/s}$$
$$F_2 = 16{,}000 \text{ N}$$

The launching angle of the second missile is determined with a trial and error method performed on computer. In the calculation, the first number used is 1.00 rad with the increment of ± 0.01. To detect whether the collision is going to take place or not, the distance between the missiles is calculated. The unsuccessful simulation terminates as the distance between them increases. When the collision is nearly occurring, finer increments for the launching angle and the time step are used.

For the present study, the increments for the final step are $\Delta\alpha = 2.0E{-7}$ and $\Delta t = 5.0E{-5}$ s. The collision condition is reached when the distance between the two missiles is less than 8 cm. The launching angle for the second missile is found to be 0.982 145 4 rad. The collision is taking place at 144.8327 s after the launching of the first missile and is 84.8327 s after the launching of the second missile. The coordinates at the collision are $X = 66.82$ km, $Z = 16.26$ km. The missile shooting missile trajectories are shown in Fig. 3.4.

2) For a spherical surface, the equations governing the motions of missiles are the same as those used in part 1. Because the trajectories of the missiles are assumed to be in the same plane, the coordinates of the first missiles are transferred using Eqs. (3.10) and (3.12) with $y = 0$. These equations are as follows:

$$X = R_0 \sin\phi + x \cos\theta \cos\phi + z \sin\phi \tag{3.19}$$

$$Z = -R_0(1 - \cos\phi) - x \cos\theta \sin\phi + z \cos\phi \tag{3.20}$$

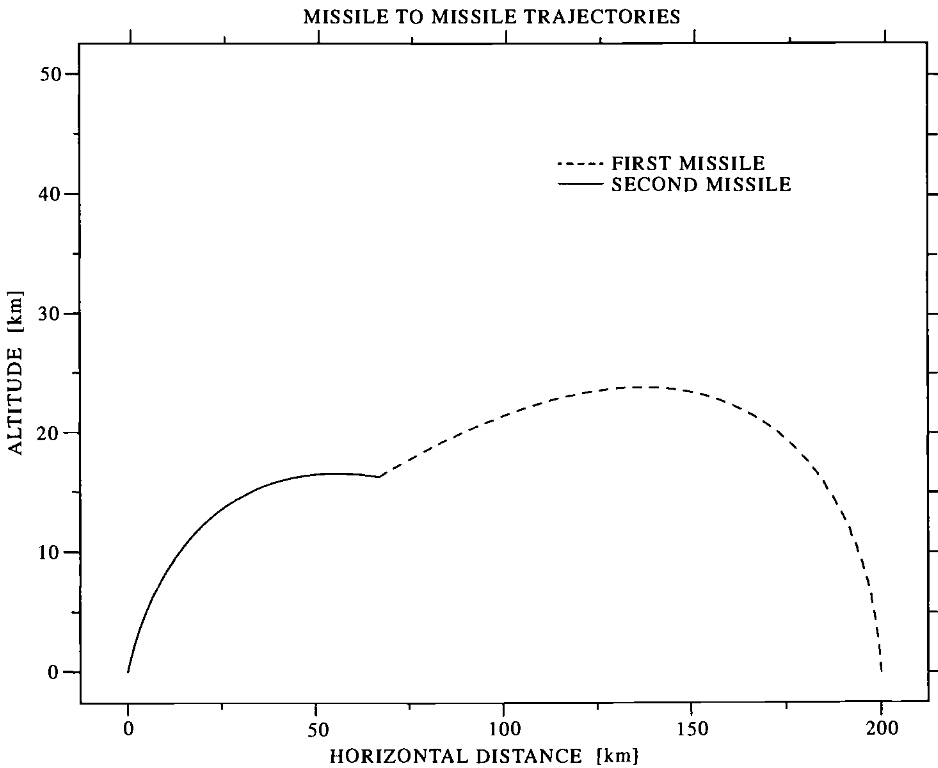

Fig. 3.4 Missile-to-missile trajectories on flat ground.

For the present case $R_0 = 6371.23$ km, $\theta = \pi$, and $\phi = 0.031391112$. Substituting these values into Eqs. (3.19) and (3.20), we have

$$X = 199{,}967.155 - 0.99950734x + 0.03138596z \qquad \text{(m)}$$

$$Z = -3138.8535 + 0.03138596x + 0.99950734z \qquad \text{(m)}$$

Note that the initial coordinates of the first missile are

$$X_0 = 199{,}967.1550 \qquad \text{(m)}$$

$$Z_0 = -3138.8535 \qquad \text{(m)}$$

The calculation procedure is the same as that used in part 1. The launching angle for the second missile is determined to be 0.9929676 rad, and the collision occurs 145.1400 s after the launching of the first missile and 85.1400 s after launching of the second missile. It is important to point out that the missiles will not collide if α is set as 0.9821454 rad, because the Earth's surface is actually spherical. The coordinates at the collision are $X = 66.64$ km and $Z = 17.18$ km. The missile

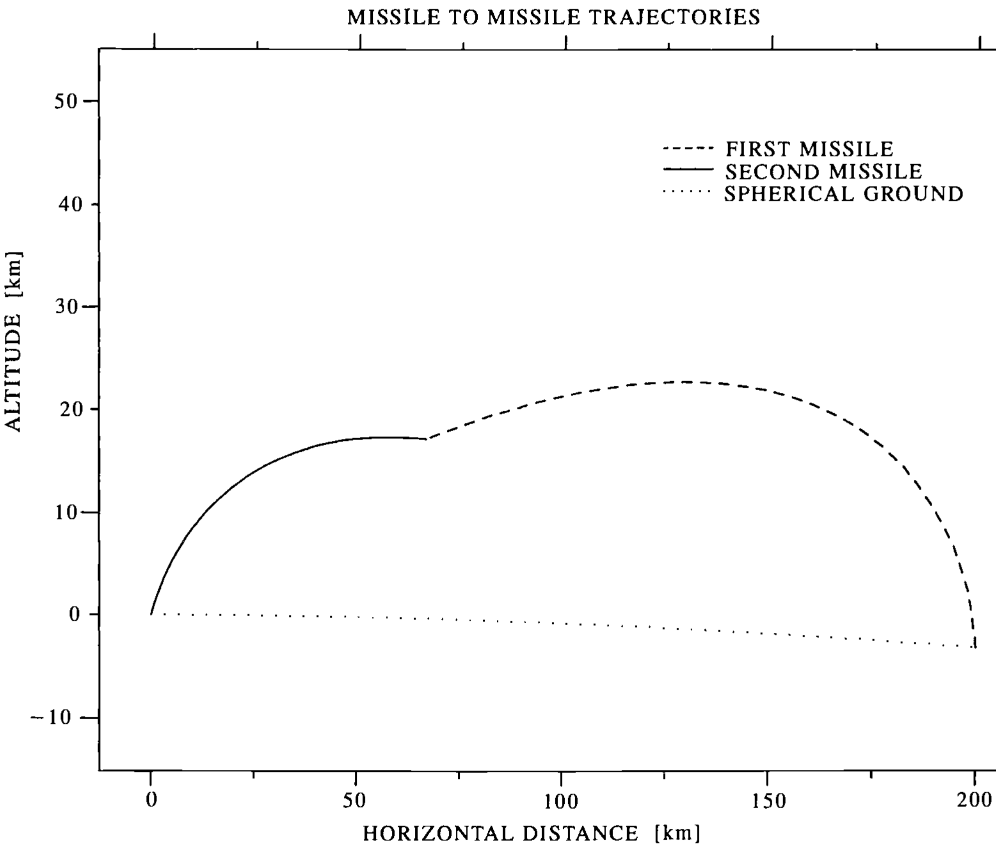

Fig. 3.5 Missile-to-missile trajectories on sphere.

trajectories are shown in Fig. 3.5. For completeness, the computer program written in Fortran is included in this section.

```
C      PROGRAM MISSILE TO MISSILE FOR EXAMPLE 3-1 ON SPHERICAL
C       SURFACE
      REAL T(18001),X1(18001),X2(18001),Z1(18001),Z2(18001),M1,M2
      OPEN (2,FILE='MSLTMSLS.FIL')
      X1(1) = 0.0
      Z1(1) = 0.0
      X = X1(1)
      Z = Z1(1)
      VX10 = 26.0472
      VZ10 = 147.7212
      M1 = 1000.0
      G = 9.81
      DM = 3.0
      VX1 = VX10
      VZ1 = VZ10
      DO 100 N = 1,18000
      VXN = VX1
      VZN = VZ1
```

```
      X2(N) = 0.0
      Z2(N) = 0.0
      AH = 0.01
      AM = M1-AH*3.0*FLOAT(N-1)
      F1 = 14500.
      F = F1
      IF (N .LT. 14513) GO TO 90
      AH = 0.00005
      AM = 564.64-AH*3.0*FLOAT(N-14512)
 90   CALL RK (X,Z,VXN,VZN,AH,AM,DM,F,G)
      X1(N+1) = X
      Z1(N+1) = Z
      VX1 = VXN
      VZ1 = VZN
100   CONTINUE
      X10 = 199967.155
      Z10 = -3138.8535
      ALP = 0.9929676
      M2 = 1000.
      V2 = 300.0
      F2 = 16000.
      F = F2
      WRITE (2,8) M1,F1,AH,M2,F2
      WRITE (2,9) X10,Z10,VX10,VZ10
      NN = 1
120   VX2 = V2*COS(ALP)
      VZ2 = V2*SIN(ALP)
      VXN = VX2
      VZN = VZ2
      WRITE (2,10) X2(1),Z2(1),VXN,VZN
      AH = 0.01
      ALPOLD = ALP
      X = X2(1)
      Z = Z2(1)
      DO 200 N = 6001,18000
      AM = M2-AH*3.0*FLOAT(N-6001)
      IF (N .LT. 14513) GO TO 150
      AH = 0.00005
      AM = 564.64-AH*3.0*FLOAT(N-14512)
150   CONTINUE
      BX0 = X
      BZ0 = Z
      CALL RK (X,Z,VXN,VZN,AH,AM,DM,F,G)
      X2(N+1) = X
      Z2(N+1) = Z
      VX2 = VXN
      VZ2 = VZN
      AX0 = X10-X1(N)*0.99950734+Z1(N)*0.03138596
      AZ0 = Z10+Z1(N)*0.99950734+X1(N)*0.03138596
      AX1 = X10-X1(N+1)*0.99950734+Z1(N+1)*0.03138596
      AZ1 = Z10+Z1(N+1)*0.99950734+X1(N+1)*0.03138596
      BX1 = X2(N+1)
      BZ1 = Z2(N+1)
      D0 = SQRT((BX0-AX0)**2+(BZ0-AZ0)**2)
      D1 = SQRT((BX1-AX1)**2+(BZ1-AZ1)**2)
      IF (D1 .GT. D0) GO TO 190
```

```
      IF (D1 .LT. 0.080) GO TO 220
      GO TO 200
190   WRITE (2,22) N,D1,D0
      GO TO 210
200   CONTINUE
210   ALP = ALPOLD-0.00000004
      NN = NN+1
      IF (NN .GT. 10) GO TO 240
      GO TO 120
220   WRITE (2,21) ALP
      WRITE (2,11)
      DO 236 I = 1,N
      AH = 0.01
      IF (I .LT. 14513) GO TO 230
      AH = 0.00005
      T(I) = 145.12+AH*FLOAT(I-14512)
      GO TO 234
230   T(I) = AH*FLOAT(I-1)
234   XX = (X10-X1(I)*0.99950734+Z1(I)*0.03138596)/1000.
      Z1(I) = (Z10+Z1(I)*0.99950734+X1(I)*0.03138596)/1000.
      X1(I) = XX
      X2(I) = X2(I)/1000.
      Z2(I) = Z2(I)/1000.
236   CONTINUE
      DO 238 I = 1,N,100
      WRITE (2,20) T(I),X1(I),Z1(I),X2(I),Z2(I),D1
238   CONTINUE
      WRITE (2,20) T(N),X1(N),Z1(N),X2(N),Z2(N),D1
      GO TO 250
240   WRITE (2,25)
  8   FORMAT ('M1 = ',F5.0,' kg  F1 = ',F6.0,' N  AH = ',F8.6,' S
     *M2 = ',F5.0,' kg  F2 = ',F6.0,' N')
  9   FORMAT (' X1o = ',F7.0,' m  Z1o = ',F8.2,' m  VX1o =
     * ',F8.2,'m/s  VZ1o = ',F8.2,' m/s ')
 10   FORMAT ('X2o = ',F7.0,' m  Z2o = ',F8.2,' m  VX2o =
     *',F8.2,'m/s  VZ2o = ',F8.2,' m/s ')
 11   FORMAT (3X,' T(s) ',6X,' X1(km)',7X,' Z1(km)',7X,'
     * X2(km)',7X,' Z2  (km)',7X, ' D1(m)')
 20   FORMAT (1X,F9.5,5(2X,E12.4))
 21   FORMAT ('MISSILES COLLIDED WITH ALPHA = ',F10.8)
 22   FORMAT ('MISSILES ARE NOT COLLIDING  N = ',I6,'  D1
     * = ',F8.4,'m  D0 = ',F8.4,'m')
 25   FORMAT ('MAXIMUM ITERATIONS EXCEEDED')
250   STOP
      END
      SUBROUTINE RK (X,Z,VXN,VZN,AH,AM,DM,F,G)
      AK1 = AH*(F/AM)*VXN/SQRT(VXN**2+VZN**2)
      BK1 = AH*((F/AM)*VZN/SQRT(VXN**2+VZN**2)-G)
      XK1 = AH*VXN
      ZK1 = AH*VZN
      AM = AM-DM*AH/2.
      AK2 = AH*(F/AM)*(VXN+AK1/2.)/SQRT((VXN+AK1/2.)**2+
     C (VZN+BK1/2.)**2)
      BK2 = AH*((F/AM)*(VZN+BK1/2.)/SQRT((VXN+AK1/2.)**2+
     C (VZN+BK1/2.)**2)-G)
```

```
      XK2 = AH*(VXN+AK1/2.)
      ZK2 = AH*(VZN+BK1/2.)
      AK3 = AH*(F/AM)*(VXN+AK2/2.)/SQRT((VXN+AK2/2.)**2+
     C (VZN+BK2/2.)**2)
      BK3 = AH*((F/AM)*(VZN+BK2/2.)/SQRT((VXN+AK2/2.)**2+
     C (VZN+BK2/2.)**2)-G)
      XK3 = AH*(VXN+AK2/2.)
      ZK3 = AH*(VZN+BK2/2.)
      AM = AM-DM*AH/2.
      AK4 = AH*(F/AM)*(VXN+AK3)/SQRT((VXN+AK3)**2+
     C (VZN+BK3)**2)
      BK4 = AH*((F/AM)*(VZN+BK3)/SQRT((VXN+AK3)**2+
     C (VZN+BK3)**2)-G)
      XK4 = AH*(VXN+AK3)
      ZK4 = AH*(VZN+BK3)
      VXN1 = VXN+(AK1+2.*AK2+2.*AK3+AK4)/6.
      VZN1 = VZN+(BK1+2.*BK2+2.*BK3+BK4)/6.
      XX = X+(XK1+2.*XK2+2.*XK3+XK4)/6.
      ZZ = Z+(ZK1+2.*ZK2+2.*ZK3+ZK4)/6.
      VXN = VXN1
      VZN = VZN1
      X = XX
      Z = ZZ
      RETURN
      END
```

3.3 General Motion of a System of Particles

Consider a system of n particles. For each particle there are two kinds of forces acting on it. One is the resultant of the external forces, and the other is the internal forces between particles. The mass of each particle is fixed. For the ith particle, the equation of motion is

$$m_i \frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} = \boldsymbol{F}_i + \sum_{\substack{j=1 \\ i \neq j}}^{n} \boldsymbol{f}_{ij} \tag{3.21}$$

where $\boldsymbol{f}_{ij}$ is the internal force exerted on the particle i by the particle j. $\boldsymbol{F}_i$ is the resultant force acting on particle i from the forces external to the system of particles. Because there are n particles in the system, the equation of motion for the system is

$$\sum_{i=1}^{n} m_i \frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} = \sum_{i=1}^{n} \boldsymbol{F}_i + \sum_{\substack{i,j=1 \\ j \neq i}}^{n} \boldsymbol{f}_{ij}$$

According to Newton's third law, the internal forces exerted by two particles i and j on each other are equal in magnitude and opposite in direction, that is $\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji}$. Therefore, the sum of the internal forces is zero and we obtain

$$\boldsymbol{F} = \sum_{i=1}^{n} m_i \frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_{i=1}^{n} m_i \boldsymbol{r}_i \tag{3.22}$$

where $\boldsymbol{F}$ is the vector sum of all the external forces acting on all the particles. To simplify this equation, let us recall the method for locating the center of mass for the system:

$$\boldsymbol{r}_c \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m_i \boldsymbol{r}_i$$

$$\boldsymbol{r}_c = \frac{\sum_{i=1}^{n} m_i \boldsymbol{r}_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M} \sum_i m_i \boldsymbol{r}_i \tag{3.23}$$

where $\boldsymbol{r}_c$ is the position vector of the center of mass. With the use of Eq. (3.23), Eq. (3.22) becomes

$$\boldsymbol{F} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} M \boldsymbol{r}_c = M \frac{\mathrm{d}^2 \boldsymbol{r}_c}{\mathrm{d}t^2} \tag{3.24}$$

Therefore, we can conclude that the motion of a system of particles is equivalent to that of a single particle with mass M located at the mass center of the system.

Now let us consider the angular momentum or the moment of momentum of a system of n particles. Taking the cross product of $\boldsymbol{r}_i$ with Eq. (3.21) leads to

$$\boldsymbol{r}_i \times m_i \frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} = \boldsymbol{r}_i \times \boldsymbol{F}_i + \boldsymbol{r}_i \times \sum_{\substack{j=1 \\ j \neq i}}^{n} \boldsymbol{f}_{ij}$$

Looking into details in the preceding equation, we find

$$\begin{aligned} \boldsymbol{r}_i \times m_i \frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} &= \boldsymbol{r}_i \times \frac{\mathrm{d}}{\mathrm{d}t}(m_i \dot{\boldsymbol{r}}_i) = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_i \times \boldsymbol{P}_i) = \frac{\mathrm{d}\boldsymbol{H}_i}{\mathrm{d}t} \end{aligned}$$

$$\dot{\boldsymbol{H}} = \sum_i \frac{\mathrm{d}\boldsymbol{H}_i}{\mathrm{d}t}$$

$$\sum_{i,j}^{n} \boldsymbol{r}_i \times \boldsymbol{f}_{ij} = 0 \qquad \text{because } \boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji}$$

Therefore, we obtain

$$\dot{\boldsymbol{H}} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = \boldsymbol{M} \tag{3.25}$$

Thus, the time rate of change of angle momentum is equal to the total moment of external forces acting on the particles with respect to a fixed point. This equation is the same as Eq. (2.32) for a single particle.

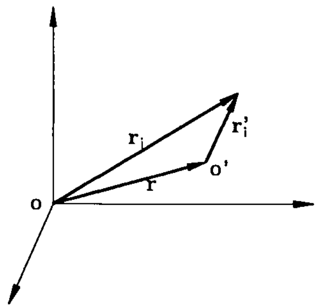

Fig. 3.6 Particles with the center of mass at 0′.

We may express Eq. (3.25) from a different perspective by considering that the center of mass is located at the origin $0'$ of another coordinate system $x'y'z'$. Then, as shown in Fig. 3.6, we have

$$\boldsymbol{r}_i = \boldsymbol{r} + \boldsymbol{r}_i'$$

where $\boldsymbol{r}$ is the position vector of the center of mass for the system of n particles, $\boldsymbol{r}_i'$ is the position vector of ith particle in $x'y'z'$. Taking the time derivative of the position vector equation, we obtain

$$\dot{\boldsymbol{r}}_i = \dot{\boldsymbol{r}} + \dot{\boldsymbol{r}}_i'$$

The angular momentum of the n particles about 0 is

$$\begin{aligned} \boldsymbol{H} &= \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{P}_i = \sum_{i=1}^{n} \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i = \sum_i (\boldsymbol{r} + \boldsymbol{r}_i') \times m_i (\dot{\boldsymbol{r}} + \dot{\boldsymbol{r}}_i') \\ &= \sum_i m_i \boldsymbol{r} \times \dot{\boldsymbol{r}} + \sum_i \boldsymbol{r} \times m_i \dot{\boldsymbol{r}}_i' + \sum_i \boldsymbol{r}_i' \times m_i \dot{\boldsymbol{r}} + \sum_i \boldsymbol{r}_i' \times m_i \dot{\boldsymbol{r}}_i' \\ &= \boldsymbol{r} \times M \dot{\boldsymbol{r}} + \sum_i \boldsymbol{r}_i' \times m_i \dot{\boldsymbol{r}}_i' \end{aligned} \tag{3.26}$$

Simplifying Eq. (3.26) is based on the fact that, because $0'$ is the center of mass, the following expressions are true:

$$\sum_i m_i \boldsymbol{r}_i' = 0,$$

$$\sum_i \boldsymbol{r} \times m_i \dot{\boldsymbol{r}}_i' = \boldsymbol{r} \times \frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i \boldsymbol{r}_i' = 0$$

$$\sum_i \boldsymbol{r}_i' \times m_i \dot{\boldsymbol{r}} = \sum_i \boldsymbol{r}_i' m_i \times \dot{\boldsymbol{r}} = 0$$

Equation (3.26) states that the angular momentum of the system with respect to point 0 equals the sum of the angular momentum of total mass M at point $0'$ with respect to 0 and the angular momentum of the system with respect to the center of mass. Furthermore, from the right hand of Eq. (3.25), we have

$$
\begin{aligned}
M &= \sum_i r_i \times F_i = \sum_i (r + r_i') \times F_i \\
&= r \times F + \sum_i r_i' \times F_i = r \times F + M'
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
M' &= \sum_i r_i' \times m_i \ddot{r}_i = \sum_i r_i' \times m_i (\ddot{r} + \ddot{r}_i') \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i r_i' \times m_i \dot{r}_i'\right] + \left(\sum_i m_i r_i'\right) \times \ddot{r} = \dot{H}'
\end{aligned}
\tag{3.28}
$$

Differentiating Eq. (3.26) with respect to time and using Eqs. (3.27) and (3.28), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H &= \frac{\mathrm{d}}{\mathrm{d}t}(r \times M\dot{r}) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i r_i' \times m_i \dot{r}_i'\right] \\
&= r \times M\ddot{r} + \dot{H}' = r \times F + M' = M
\end{aligned}
\tag{3.29}
$$

From Eq. (3.29), we see that the total moment acting on the system with respect to point 0 equals the sum of the moment produced by the total external force with respect to point 0 and the moment of the system with respect to point 0′.

3.4 Gravitational Force and Potential Energy

When a point mass is in the vicinity of a large mass, such as Earth, it experiences a gravitational force directed toward the mass center of the large mass and possesses a potential energy with respect to the large mass. If there are two point masses placed side by side, an attractive force will exist between them. According to Newton's law of universal gravitation, the magnitude of this attractive force can be expressed as

$$
F = G\frac{m_1 m_2}{r^2}
$$

where G is the universal gravitational constant and is 6.67×10^{-11} N-m^2/kg^2, and r is the distance between the two point masses m_1 and m_2. Suppose that the position vector of m_1 is r_1 and that of m_2 is r_2 as shown in Fig. 3.7. Then the force acting on m_1 is

$$
F_1 = G\frac{m_1 m_2}{|r_2 - r_1|^3}(r_2 - r_1)
$$

When m_2 is a distributed mass, however, the gravitational force on m_1 becomes

$$
F_1 = m_1 \int_v G\frac{\rho(r')\mathrm{d}v'}{|r' - r|^3}(r' - r)
$$

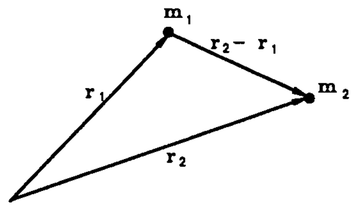

Fig. 3.7 Relationship between two point masses.

where $\rho(r')$ is the density of the distributed mass. The relationship between m_1 and a distributed mass is shown in Fig. 3.8. Rearranging the preceding equation gives the gravitational intensity as

$$\frac{F_1}{m_1} = g = \int_v G \frac{\rho(r')\,\mathrm{d}v'}{|r' - r|^3}(r' - r) \tag{3.30}$$

Gravitational force is a typical conservative force that can be expressed as

$$g = -\nabla V \tag{3.31}$$

where V = potential energy per unit mass or gravitational potential. In terms of the gradient of a scalar function, Eq. (3.30) can be written as

$$g = G \int_v \nabla\left(\frac{1}{|r' - r|}\right)\rho\,\mathrm{d}v' \tag{3.32}$$

Therefore, the gravitational potential is

$$V = -G \int_v \frac{\rho\,\mathrm{d}v'}{|r' - r|} \tag{3.33}$$

Example 3.2

Derive expressions for the gravitational force and the potential energy for a point mass under the following two circumstances: 1) outside a uniform solid sphere and 2) inside the solid sphere.

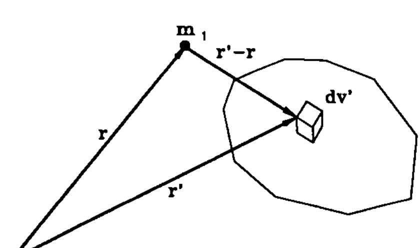

Fig. 3.8 Relationship between m_1 and a distributed mass.

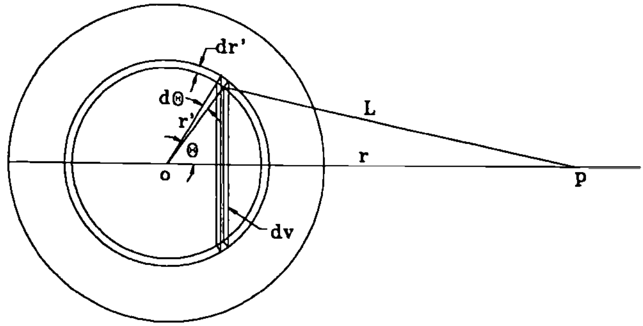

Fig. 3.9 Point P located outside a uniform solid sphere.

Solution. 1) Outside a uniform solid sphere, consider that a particle with unit mass is located at P; the distance between the center of the sphere and the particle is r as shown in Fig. 3.9. The infinitesimal volume under consideration is

$$\mathrm{d}v = (2\pi r'^2 \sin\theta)\mathrm{d}\theta\,\mathrm{d}r'$$

The distance between p and $\mathrm{d}v$ is

$$L = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$$

Using Eq. (3.33), the potential energy is

$$\begin{aligned} V &= -G\rho\int_v \frac{\mathrm{d}v}{L} = -G\rho\int_0^R\int_0^\pi \frac{2\pi r'^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r'}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} \\ &= -\frac{G\rho}{r}\frac{4\pi}{3}R^3 = -\frac{GM}{r} \end{aligned} \tag{3.34}$$

where M is the mass of the solid sphere. This result states that the potential of unit mass outside a solid sphere is equivalent to that of a point mass with the same mass concentrated at the center of the sphere. From the result of Eq. (3.34), we find the gravitational force as

$$\boldsymbol{g} = -\nabla V = \boldsymbol{e}_r\frac{\partial}{\partial \boldsymbol{r}}\left(\frac{GM}{r}\right) = -\boldsymbol{e}_r\frac{GM}{r^2} \tag{3.35}$$

where $\boldsymbol{e}_r$ is the unit vector along r. This expression is used for calculating the gravitational acceleration.

2) Inside a uniform solid sphere, consider that the point mass is at P located inside the sphere. The infinitesimal volume is a ring and can be expressed as

$$\mathrm{d}v = (2\pi r'\sin\theta)\mathrm{d}r'(r'\mathrm{d}\theta)$$

The distance between P and $\mathrm{d}v$ is L:

$$L = [(r'\sin\theta)^2 + (R\cos\theta_1 - r'\cos\theta)^2]^{1/2}$$

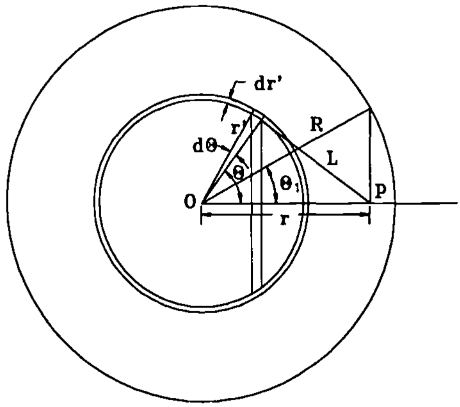

Fig. 3.10 Point P located inside a uniform solid sphere.

Therefore, the potential energy is

$$V = -G\rho \int_0^R \int_0^\pi \frac{2\pi r'^2 \sin\theta \, d\theta \, dr'}{\sqrt{r'^2 + R^2\cos^2\theta_1 - 2Rr'\cos\theta_1\cos\theta}}$$

$$= -G\rho\frac{2\pi}{3}\left[3R^2 - (R\cos\theta_1)^2\right] = -\frac{2}{3}\pi G\rho(3R^2 - r^2) \qquad (3.36)$$

where r is the distance of op. In the integration, it ought to be pointed out that when the integrand is integrated with respect to θ, the term for the lower integral limit is always kept to be positive so that the result is the subtraction of the upper limit term by the lower limit term. And the gravitational intensity is found:

$$g = -\nabla V = -\frac{4\pi}{3}G\rho r \boldsymbol{e}_r \qquad (3.37)$$

Note that the g approaches zero as r reaches zero and the potential energy reaches minimum at the center of the sphere.

Example 3.3

Suppose that a homogeneous right circular cylinder of radius R, height L, and mass M is placed along the z axis between $z = 0$ and $z = L$ as shown in Fig. 3.11. Find the gravitational intensity and potential of the cylinder on the axis at distance h from the origin with $h > L$.

Solution. Consider that the infinitesimal element in the cylinder is a ring with the cross section area of $dr\,dz$ and with

$$dv = 2\pi r \, dr \, dz$$

The distance from the ring to point P is

$$S = \sqrt{r^2 + (h-z)^2}$$

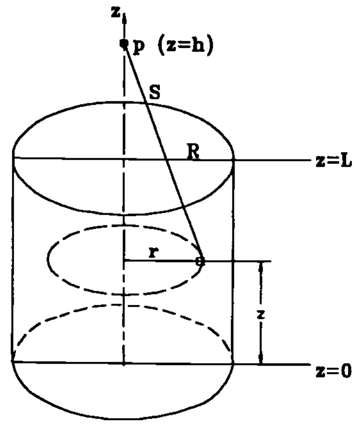

Fig. 3.11 Point P located on the axis of the cylinder.

Hence the gravitational potential is

$$
\begin{aligned}
V &= -G\rho \int_0^L \int_0^R \frac{2\pi r\,\mathrm{d}r\,\mathrm{d}z}{\sqrt{r^2 + (h-z)^2}} \\
&= -2\pi G\rho \int_0^L [r^2 + (h-z)^2]^{1/2}|_0^R\,\mathrm{d}z \\
&= -2\pi G\rho \int_0^L \{[R^2 + (h-z)^2]^{1/2} - (h-z)\}\mathrm{d}z \\
&= \pi G\rho \Bigg\{ [(h-L)\sqrt{(h-L)^2 + R^2} - h\sqrt{h^2 + R^2}] \\
&\quad + [h^2 - (h-L)^2] + R^2 \ln \left[\frac{(h-L) + \sqrt{(h-L)^2 + R^2}}{h + \sqrt{h^2 + R^2}} \right] \Bigg\}
\end{aligned}
$$

And the gravitational intensity is

$$
\begin{aligned}
g &= -\nabla V = -k\frac{\partial}{\partial h} V \\
&= -k2\pi G\rho\left[\sqrt{(h-L)^2 + R^2} - \sqrt{h^2 + R^2} + L\right]
\end{aligned}
$$

3.5 Collision of Two Spheres on a Plane

The action of two bodies colliding with a large inertial force in a short time interval is called impact. Depending upon the material properties of the bodies, collision can be elastic or inelastic. We shall discuss these two kinds of collision in this section.

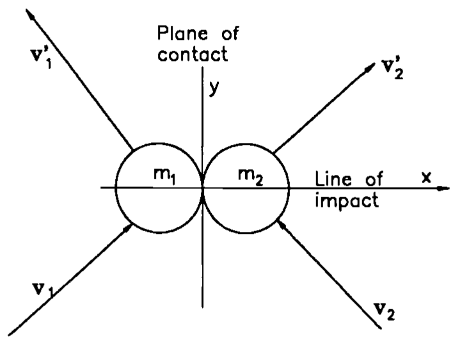

Fig. 3.12 Oblique central impact.

For the present study, the bodies in collision are modeled as two spheres with identical diameter and with the centers of mass at the centers of spheres. This, however, does not imply that the mass must be the same. The system may be pictured as two balls of different mass colliding on a frictionless table, which forms a perfect plane perpendicular to the gravitational force. For a general two-dimensional case, the collision is named an oblique central impact. The plane tangent to two spheres at the contact point is called the plane of contact. The line perpendicular to the plane of contact is termed the line of impact that goes through two centers of spheres. For an oblique central impact, the velocities of spheres are at angles away from the line of impact as shown in Fig. 3.12. When the velocities are on the line of impact, the action is called central impact. Therefore, a central impact is a special case of oblique central impact. The method of analysis for oblique central impact can be applied easily to the central impact.

Automobile accidents are common occurrences in this country. Every day there are thousands of car collisions with hundreds of injuries and deaths. As we study the collision of bodies, it is interesting to try to answer two questions arising from car collisions. In a collision, is the driver of a heavier car safer than the driver of a lighter car? As an unavoidable head-on collision is about to happen, should the drivers accelerate their cars as much as possible to protect themselves? These questions will be answered in the examples. Although cars are in complicated shapes, the modeling of cars as spheres is only the first step in studying car collisions.

Certainly the application of the collision of two spheres is not limited to billiards and automobiles. It can be applied also to the collision of molecules in chemical reactions or in turbulent flows. Let us study the collisions in two different conditions, elastic and inelastic, as follows.

Elastic Collision

During the collision there is an impulsive force between two masses. If the force is not very large, the stresses developed on the spheres are below the yielding points. Then the shapes of the two spheres are restored completely to their original forms without any permanent deformation as shown in Fig. 3.13. Such a collision is called an elastic collision.

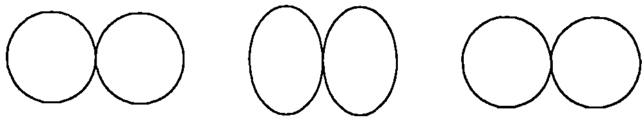

Fig. 3.13 Deformation and restitution during elastic impact.

Because there is no external force involved during collision, the total momentum of the two spheres is conserved, and we have

$$m_1 \boldsymbol{V}_1 + m_2 \boldsymbol{V}_2 = m_1 \boldsymbol{V}_1' + m_2 \boldsymbol{V}_2' \tag{3.38}$$

where $\boldsymbol{V}_i$ and $\boldsymbol{V}_i'$ are velocities of m_i before and after impact respectively.

For the elastic collision, because the shapes of the spheres are completely restored, the kinetic energy of the system is conserved and we have

$$\tfrac{1}{2} m_1 V_1^2 + \tfrac{1}{2} m_2 V_2^2 = \tfrac{1}{2} m_1 V_1'^2 + \tfrac{1}{2} m_2 V_2'^2 \tag{3.39}$$

Equation (3.38) is a vector equation that may be considered as two equations in terms of x and y directions. Hence there are three equations, but, in general, there are four unknowns: V_{1x}', V_{1y}', V_{2x}', V_{2y}', the velocity components after the collision. To determine them, additional conditions must be specified. In the collision process, no coordinate system exists in the space. Without loss of generality, we choose x axis along the line of impact and y axis along the plane of impact. With the frictionless model, it is reasonable to accept that the velocity components in the y direction are not changed, i.e.,

$$V_{1y}' = V_{1y}$$

$$V_{2y}' = V_{2y}$$

Then the momentum and energy equations become

$$m_1 V_{1x} + m_2 V_{2x} = m_1 V_{1x}' + m_2 V_{2x}' \tag{3.40}$$

$$m_1 V_{1x}^2 + m_2 V_{2x}^2 = m_1 V_{1x}'^2 + m_2 V_{2x}'^2 \tag{3.41}$$

Now V_{1x}' and V_{2x}' can be determined by Eqs. (3.40) and (3.41).

Inelastic Collision

During collision, sometimes the stresses produced by the impact force are much higher than the yielding strength of the materials, and permanent deformation results, as shown in Fig. 3.14. Such a collision is called an inelastic collision. With permanent deformations, some energy is dissipated by the stress-strain energy so that the conservation of energy is no longer true. Therefore, additional information will be needed to predict the velocities after the impact. To do this, we will define

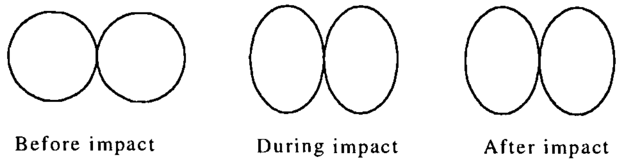

Fig. 3.14 Deformation and restitution during inelastic impact.

the coefficient of restitution ϵ, as the ratio of the impulse during the restitution period to the impulse during the deformation period, i.e.,

$$\epsilon = \frac{\text{impulse during restitution}}{\text{impulse during deformation}} = \frac{\int R\,\mathrm{d}t}{\int D\,\mathrm{d}t} \tag{3.42}$$

where R and D are the impact forces during restitution and deformation periods, respectively. The deformation period is the interval between the beginning of contact of the spheres and the instant of the maximum deformation, and the restitution period is the interval between the instant of maximum deformation and the moment that the spheres just separate. Thus, the changes of momentum of m_1 in these periods can be written as

$$\int D\,\mathrm{d}t = -[(m_1 V_{1x}) - (m_1 V_{1x})_D]$$

$$\int R\,\mathrm{d}t = -[(m_1 V_{1x})_D - (m_1 V'_{1x})]$$

Therefore,

$$\epsilon = \frac{(V_{1x})_D - V'_{1x}}{V_{1x} - (V_{1x})_D} \tag{3.43}$$

where $(V_{1x})_D$ is the velocity component in the x direction of m_1 at the maximum deformation. Similarly, for mass m_2,

$$\epsilon = \frac{(V_{2x})_D - V'_{2x}}{V_{2x} - (V_{2x})_D} \tag{3.44}$$

Note that, at the moment of maximum deformation, the two masses are in contact, and their velocities are the same along the line of impact, $(V_{1x})_D = (V_{2x})_D$. Thus Eqs. (3.43) and (3.44) can be combined to become

$$\epsilon = -\frac{V'_{2x} - V'_{1x}}{V_{2x} - V_{1x}} \tag{3.45}$$

The values of ϵ presumably are known for some common materials. With the use of the momentum equation in x direction, Eq. (3.40) together with Eq. (3.45), V'_{1x} and V'_{2x} can be predicted.

The following are a few remarks about the significance of the coefficient of restitution. During a perfectly elastic collision, the impulse for the period of restitution equals the impulse for the period of deformation, so that the coefficient of restitution is unity for this case. For inelastic collisions, the coefficient of restitution is less than unity because the impulse is diminished on restitution as a result of failure of the spheres to resume their original shapes. For a perfectly plastic collision, $\epsilon = 0$ (i.e., $V'_{2x} = V'_{1x}$) and the spheres remain in contact.

Example 3.4

Two billiard balls of the same size and mass collide with the velocities of approach shown in Fig. 3.15. What are the final velocities of the balls directly after an elastic collision?

Solution. The initial velocities of the balls are

$$V_1 = 5i \text{ m/s}$$

$$V_2 = -7.07i + 7.07j \text{ m/s}$$

Because there is no friction, the velocities after the impact in y direction are

$$V'_{1y} = 0$$

$$V'_{2y} = 7.07 \text{ m/s}$$

With $m_1 = m_2$, the momentum equation in the x direction gives

$$\begin{aligned} V_{1x} + V_{2x} &= V'_{1x} + V'_{2x} \\ 5 + (-7.07) &= V'_{1x} + V'_{2x} = -2.07 \end{aligned} \tag{3.46}$$

The energy equation (3.41) leads to

$$V_{1x}^2 + V_{2x}^2 = V_{1x}'^2 + V_{2x}'^2 \tag{3.47}$$

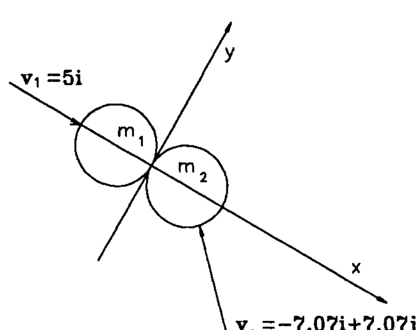

Fig. 3.15 Initial condition of the impact of balls.

Solving Eq. (3.46) and Eq. (3.47) simultaneously leads to

$$V'_{1x} - V'_{2x} = V_{2x} - V_{1x} = -7.07 - 5.0 = -12.07$$

Therefore, in the x direction, the velocities after the impact are

$$V'_{1x} = -7.07 \text{ m/s}$$

$$V'_{2x} = 5.00 \text{ m/s}$$

In the vector form, the velocities after the impact are

$$\boldsymbol{V}'_1 = -7.07\boldsymbol{i} \text{ m/s}$$

$$\boldsymbol{V}'_2 = 5.0\boldsymbol{i} + 7.07\boldsymbol{j} \text{ m/s}$$

Example 3.5

Prove that in a case of a two-car, head-on collision, the driver of a heavier car is usually less severely injured.

Solution. Let m_1, m_2 represent the mass of the two cars. Assume that car 2 is heavier than car 1, i.e., $m_2 > m_1$, and the collision is elastic. Then, from the momentum equation, we have

$$m_1(V'_1 - V_1) = m_2(V_2 - V'_2)$$

$$m_1 \Delta V_1 = m_2 \Delta V_2$$

The preceding result says that the change of momentum of car 1 equals that of car 2. Becasue $m_2 > m_1$, we conclude $|\Delta V_2| < |\Delta V_1|$, that is, the change of velocity for car 2 is less than for car 1.

Let m_d be the mass of the driver, and Δt be the time interval of the impact. Assume that the drivers have the same mass. Thus the inertial force acting on the driver is

$$m_d \frac{\Delta v}{\Delta t}$$

Comparing the inertial force acting on the two drivers, we have

$$m_d \left| \frac{\Delta V_2}{\Delta t} \right| < m_d \left| \frac{\Delta V_1}{\Delta t} \right| \text{ as } m_2 > m_1$$

Because the inertial force on the driver in car 2 is less than that on the driver in car 1, the injury to the driver in a heavier car is less than that in the lighter car.

Example 3.6

Estimate the difference in impact force for the following two cases: 1) Two cars have the same constant velocity of 50 mph but in opposite direction, and 2) one of the cars is accelerating at 5 ft/s^2 although at the time of collision the cars' velocities are the same as case 1. The mass of the cars are 100 slugs, and the duration of impact is 0.020 s. The two cars are stopped after the collision.

Solution. 1) Let F be the impact force

$$F\,\Delta t = m\Delta V$$

$$m = 100\,\text{slug}$$

Because the cars are stopped after the collision, their final velocities are zero. Therefore,

$$\Delta V = \frac{50 \times 5280}{3600} = 73.5\,\text{ft/s}$$

$$F = m\frac{\Delta V}{\Delta t} = 100\frac{73.5}{0.020} = 367.5 \times 10^3\,\text{lbf}$$

2) With the acceleration in one car, additional external force due to friction must be considered. The total impact force is

$$F' = m\frac{\Delta V}{\Delta t} + F_f$$

However,

$$F_f = ma = 100 \times 5 = 500\,\text{lbf}$$

$$F' = 367.5 \times 10^3 + 0.5 \times 10^3 = 368.0 \times 10^3\,\text{lbf}$$

The result shows that the impact force due to the acceleration of one car is very small compared with the total impact force.

Problems

3.1. Find the transformation of coordinates for the trajectory of the enemy missile. The enemy's missile site is 1000 km away from ours and is on a mountain 5 km above the surface of the average radius of the Earth. Assume that for the missile-to-missile collision, two trajectories are contained in the same plane.

3.2. Consider that the gravitational force always is pointing toward the center of the Earth. Suppose that the enemy's missile is launched from the site as given in Problem 3.1. What are the components of the gravitational force in the (x, z) directions?

3.3. Suppose that a rocket is launched vertically, and at the time of burnout the speed of the rocket is v_0 at the altitude of h_0 above the surface of the Earth. Use the expression $g = k/r^2$ for the gravitational acceleration, where k is a constant and r is the distance from the center of Earth to the rocket. Find the maximum height the rocket can reach. Also find the escape velocity for a rocket launched in a vertical position.

3.4. Show that the gravitational attraction due to a homogeneous circular disk at a point on the axis of the disk is

$$\frac{2MG}{a^2}\left[1 - \frac{h}{\sqrt{h^2 + a^2}}\right]$$

where M is the mass of the disk, a is the radius of the disk, and h is the height of the point above the center of the disk.

3.5. A uniform sphere of mass M is embedded in a hole of radius R in an infinite thin plane having mass per unit area σ. Find the gravitational field intensity and the potential energy per unit mass at a distance d above the center of the sphere.

3.6. In introductory dynamics, the potential energy of a mass m at z above the ground is always expressed as mgz. Now we have learned that the potential energy of mass m outside the spherical Earth is $-GmM/r$. What is the relationship between them?

3.7. Explain that, in the oblique impact, the coefficient of restitution cannot be defined in the direction that is not perpendicular to the plane of contact.

3.8. Two spherical balls of the same size and mass are in a head-on collision. Because of a manufacturing defect, the center of mass of one ball is not at the center of the sphere. Formulate the equations governing this impact. Predict the motions of the balls after the impact.

3.9. Suppose that a hard, small ball m drops vertically at a point on a hard, solid spherical surface as shown in Fig. P3.9, with mass $M \gg m$. The initial height of the ball is h_0.

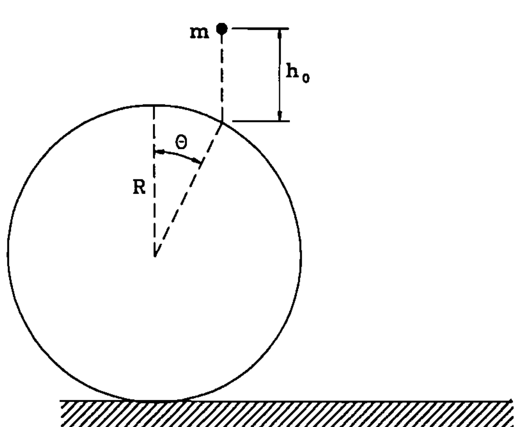

Fig. P3.9

(a) What is the velocity of the ball immediately after the impact for a coefficient of restitution $e = 0.85$?

(b) What is the trajectory of the ball after the impact but before it lands on the floor?

3.10. A ball is dropped from a height of 3 m onto a level floor. If the coefficient of restitution $e = 0.9$, how long will it take the ball to come to rest? What is the total distance traveled by the ball?

4
Lagrange's Equations and the Variational Principle

FUNDAMENTAL equations in dynamics are based on Newton's second law of motion. When Newton's law is used to formulate a problem, an explicit expression of force or torque is required. Such expression may not be easy to obtain. An alternative approach is to employ Lagrange's equations. In the Lagrangian formulation for conservative systems, expressions for kinetic and potential energies are required, but knowledge of the force or torque is not needed.

There are different forms of Lagrange's equations. One form is for dynamic systems without constraints between generalized coordinates, which are coordinates based on configurations of the systems and are discussed in the next section. Another form is for systems with constraints. In this form, constraint relations are incorporated directly into Lagrange's equations as Lagrangian multipliers and constraint forces.

The Hamilton equations are discussed in Section 4.3. These equations are parallel to the Lagrangian equations for systems without constraints. Through this parallel approach, readers will become more familiar with the Lagrangian equations. The general form of Lagrangian equation is studied in Section 4.4. Different constraints are discussed, and Lagrangian multipliers are introduced for solving the problems. Note that Lagrangian multipliers are related to nonconservative forces. Many examples are given in this section.

In Section 4.5, the variational principle is introduced. The purpose of this principle is for optimization. It is discussed here because Lagrange's equations can be derived from the optimization of the Lagrangian function of dynamic systems. A case of optimum with a constraint condition also is studied. Examples are given for the application of the variational principle.

4.1 Generalized Coordinates, Velocities, and Forces

Generalized coordinates are the coordinates that must be specified in order to describe the configuration of a system. If a system of N particles is under consideration, three coordinates are needed to specify the position of one particle so that $3N$ coordinates are required for N particles. The system is said to have $3N$ degrees of freedom. If some coordinates are related by j equations or constraints, the degrees of freedom are reduced to $3N - j$.

For a particle traveling along a straight line, the only coordinate needed is the particle's traveling distance. For a wheel rotating on its fixed shaft, the coordinate describing the wheel is the rotating angular displacement. For a wheel with a shaft moving along a straight line, two coordinates must be specified: the distance traveled by the shaft and the angular displacement of the wheel. For a pair of long-nosed pliers lying on a table, four coordinates are needed to describe the system: (x, y) coordinates for the location of the center of pivot, α for the angle between the surface of the first jaw and the x axis, and β for the angle between the

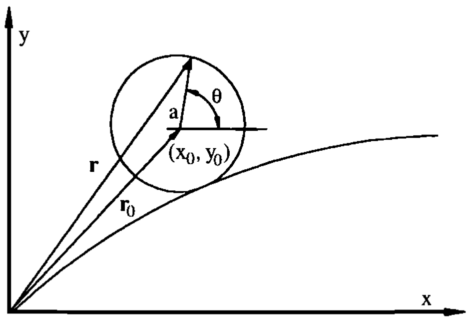

Fig. 4.1 Wheel rolling on a curved ground.

surfaces of two jaws. Because of the nature of generalized coordinates, the number of such coordinates is called the number of degrees of freedom of the system.

Usually, symbols $(q_1, q_2, \ldots, q_n)$ are used for generalized coordinates. A position vector $\boldsymbol{r}$ always can be expressed as a function of q, and we may write

$$\boldsymbol{r} = \boldsymbol{r}(q_1, q_2, \ldots, q_n)$$

or

$$\boldsymbol{r} = \boldsymbol{r}(q) \tag{4.1}$$

To illustrate the preceding statement, let us consider a point at the edge of a wheel rolling without slipping on a curved ground as shown in Fig. 4.1.

$$\boldsymbol{r} = \boldsymbol{r}_0 + a(\cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j}) = (x_0 + a\cos\theta)\boldsymbol{i}$$

$$+ (y_0 + a\sin\theta)\boldsymbol{j} = \boldsymbol{r}(x_0, y_0, \theta) = \boldsymbol{r}(q_1, q_2, q_3)$$

where $q_1 = x_0$, $q_2 = y_0$, and $q_3 = \theta$. As the particle moves, we have

$$\dot{\boldsymbol{r}} = \sum_{\rho=1}^{n} \frac{\partial \boldsymbol{r}}{\partial q_\rho} \dot{q}_\rho \tag{4.2}$$

The quantities $\dot{q}_\rho \equiv \mathrm{d}q_\rho/\mathrm{d}t$ are called generalized velocities. Equation (4.2) suggests that

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}(q, \dot{q}) \tag{4.3}$$

Here q and $\dot{q}$ are considered independent variables.

Furthermore, a typical force F acts at a point (x, y, z). The virtual work produced by the force is

$$\delta W = \boldsymbol{F} \cdot \delta \boldsymbol{r} \tag{4.4}$$

where $\delta \boldsymbol{r}$ is the virtual displacement and can be expressed in terms of generalized coordinates as

$$\delta \boldsymbol{r} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}}{\partial q_i} \delta q_i \tag{4.5}$$

With the use of Eq. (4.5), Eq. (4.4) becomes

$$\delta W = \sum_{i=1}^{n} \left(\boldsymbol{F} \cdot \frac{\partial \boldsymbol{r}}{\partial q_i} \delta q_i \right) = \sum_{i=1}^{n} (Q_i \delta q_i) \tag{4.6}$$

where $Q_i \equiv \boldsymbol{F} \cdot (\partial \boldsymbol{r} / \partial q_i) \equiv$ generalized force.

For a conservative force as defined in Section 2.5,

$$\boldsymbol{F} = -\nabla V$$

$$\boldsymbol{F} \cdot \frac{\partial \boldsymbol{r}}{\partial q_i} = -\nabla V \cdot \frac{\partial \boldsymbol{r}}{\partial q_i} = -\frac{\partial V}{\partial q_i}$$

Hence,

$$Q_i = -\frac{\partial V}{\partial q_i} \qquad i = 1, 2, \ldots, n \tag{4.7}$$

The generalized forces for a conservative system are the arithematic inverse of the partial derivatives of potential energy with respect to the generalized coordinates.

4.2 Lagrangian Equations

Consider a system of N particles with n degrees of freedom. A position vector $\boldsymbol{r}_i$ for the position of ith particle is, in general, a function of generalized coordinates and time.

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, \ldots, q_n, t) = \boldsymbol{r}_i(q, t) \tag{4.8}$$

where q represents all the various q. In Eq. (4.8) q and t are independent variables, and the velocity of the ith particle is

$$\boldsymbol{v}_i = \boldsymbol{v}_i(q, \dot{q}, t) \tag{4.9}$$

where $\dot{q}$ is the generalized velocity representing $(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$. Certainly,

$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \boldsymbol{v}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t} \tag{4.10}$$

On the other hand, considering a virtual displacement

$$\delta \boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j \tag{4.11}$$

Note that the symbol δ is used for virtual displacement. No time is needed to reach $\delta \boldsymbol{r}_i$. Taking the partial derivative of $\boldsymbol{v}_i$ with respect to generalized velocity $\dot{q}_k$ from Eq. (4.10) gives

$$\frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k} \left[\sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t} \right] = \frac{\partial \boldsymbol{r}_i}{\partial q_k} \tag{4.12}$$

Here we find that the partial derivative of the velocity of ith particle with respect to $\dot{q}_k$ equals the partial derivative of the position vector with respect to q_k. Differentiating $(\partial \boldsymbol{r}_i/\partial q_k)$ with respect to time yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right) = \sum_{j=1}^{n} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_k \partial q_j}\dot{q}_j + \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right) \tag{4.13}$$

Taking the partial derivative of $\dot{\boldsymbol{r}}_i$ with respect to q_k from Eq. (4.10), we have

$$\begin{aligned} \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_k} &= \sum_{j=1}^{n} \frac{\partial}{\partial q_k}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j\right) + \frac{\partial}{\partial q_k}\left(\frac{\partial \boldsymbol{r}_i}{\partial t}\right) \\ &= \sum_{j=1}^{n}\left(\frac{\partial^2 \boldsymbol{r}_i}{\partial q_k \partial q_j}\dot{q}_j\right) + \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right) \end{aligned} \tag{4.14}$$

Comparing Eq. (4.13) to Eq. (4.14), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right) = \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_k} \tag{4.15}$$

Now let us consider D'Alembert's principle for the ith particle of the system of N particles:

$$\boldsymbol{F}_i - \dot{\boldsymbol{P}}_i = 0 \tag{4.16}$$

where $\dot{\boldsymbol{P}}_i$ is the rate change of momentum of the ith particle. In addition, let us imagine a virtual displacement of $\delta \boldsymbol{r}_i$ for the ith particle. For the system we have

$$\sum_{i=1}^{N}(\boldsymbol{F}_i - \dot{\boldsymbol{P}}_i)\cdot \delta \boldsymbol{r}_i = 0 \tag{4.17}$$

Note that Eq. (4.17) is equivalent to Eq. (1.34). When D'Alembert's principle is considered, the inertia force is one of the applied forces. In Section 1.6, we reached the conclusion that the virtual work of applied forces in equilibrium is zero. Now, let us separately examine the two terms in detail as follows:

$$\sum_{i=1}^{N}\boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = \sum_{i=1}^{N}\sum_{j=1}^{n}\boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n} Q_j \delta q_j \tag{4.18}$$

where

$$Q_j = \sum_{i=1}^{N}\boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \tag{4.19}$$

Q_j is the generalized force, and

$$\begin{aligned} \sum_{i=1}^{N}\dot{\boldsymbol{P}}_i \cdot \delta \boldsymbol{r}_i &= \sum_{i=1}^{N}\sum_{j=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}(m_i \boldsymbol{\nu}_i)\cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}\delta q_j \\ &= \sum_{i,j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \boldsymbol{\nu}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) - m_i \boldsymbol{\nu}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right)\right]\delta q_j \end{aligned}$$

Using Eqs. (4.12) and (4.15), we obtain

$$\sum_{i=1}^{N}\dot{\boldsymbol{P}}\cdot\delta\boldsymbol{r}_i = \sum_{i,j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i\boldsymbol{\nu}_i\cdot\frac{\partial\boldsymbol{\nu}_i}{\partial\dot{q}_j}\right) - m_i\boldsymbol{\nu}_i\cdot\frac{\partial\boldsymbol{\nu}_i}{\partial q_j}\right]\delta q_j$$

$$= \sum_j\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j \tag{4.20}$$

where $T = \sum_{i=1}^{N}\frac{1}{2}m_i v_i^2$ = kinetic energy of the system. Combining Eqs. (4.18) and (4.20), we find that

$$\sum_j\left[Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right) + \frac{\partial T}{\partial q_j}\right]\delta q_j = 0 \tag{4.21}$$

Because all q_j are independent, the terms in the brackets must be zero, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \tag{4.22}$$

This is the first form of Lagrange's equations. For a conservative system,

$$Q_j = \sum_{i=1}^{N}\boldsymbol{F}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_j} = -\sum_{i=1}^{N}(\nabla V)_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_i} = -\frac{\partial V}{\partial q_j}$$

where V is the potential energy of the system and is a function of generalized coordinates only. Now Eq. (4.22) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right) - \frac{\partial(T-V)}{\partial q_j} = 0$$

Because potential energy is not a function of generalized velocity,

$$\frac{\partial V}{\partial\dot{q}_j} = 0$$

which can be subtracted from the first term. Thus, the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad j = 1, 2, \ldots, n \tag{4.23}$$

where the Lagrangian function $L = T - V$. Equation (4.23) is Lagrange's equation for a conservative system in which L is, in general, a function of q, $\dot{q}$, and t. For a nonconservative system, the generalized force can be expressed as a combination of conservative and nonconservative forces.

$$Q_j = -\frac{\partial V}{\partial q_j} + \mathcal{F}_j$$

where $\mathcal{F}_i$ is the nonconservative force. Therefore, in general, Lagrange's equation is in the form of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \mathcal{F}_j \qquad j = 1, 2, \ldots, n \tag{4.24}$$

Example 4.1

Find the differential equation of motion for a simple pendulum of length L and finite angle of θ measured from the vertical as shown in Fig. 4.2.

Solution. Because the angle θ is sufficient to describe the configuration of the system, it is used as the generalized coordinate, and the system has only one degree of freedom.

Kinetic energy:

$$T = \tfrac{1}{2}m(L\dot{\theta}^2)$$

Potential energy:

$$V = mgL(1 - \cos\theta)$$

Lagrangian function:

$$L = T - V = \tfrac{1}{2}m(L\dot{\theta})^2 - mgL(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = mL^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgL\sin\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = mL^2\ddot{\theta} + mgL\sin\theta = 0$$

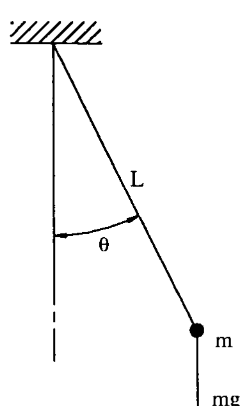

Fig. 4.2 Simple pendulum.

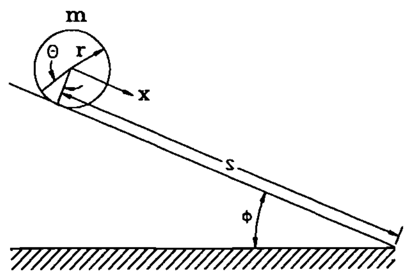

Fig. 4.3 Hoop rolling down an inclined plane.

Hence, the equation of motion is

$$\ddot{\theta} + (g/L)\sin\theta = 0 \tag{4.25}$$

Example 4.2

A hoop of radius r and mass m is rolling, without slipping, down an inclined plane at an angle ϕ. Find the equation of motion.

Solution. For the generalized coordinates, we choose the angle of rotation of the hoop θ and the distance x traveled by the center of the hoop.

Kinetic energy:

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2$$

where I is the mass moment of inertia of the hoop. Because $\dot{x} = r\dot{\theta}$ and $I = mr^2$, the kinetic energy, potential energy, and Lagrangian function of the hoop are

$$T = \tfrac{1}{2}m(r\dot{\theta})^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 = m(r\dot{\theta})^2$$

$$V = mg(s - x)\sin\phi = mg(s - r\theta)\sin\phi$$

$$L = T - V = m(r\dot{\theta})^2 - mg\,\sin\phi(s - r\theta)$$

Applying Lagrange's equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 2mr^2\ddot{\theta} - mgr\,\sin\phi = 0$$

Hence, the equation of motion is

$$\ddot{\theta} = (1/2r)g\,\sin\phi \tag{4.26}$$

Example 4.3

Two simple pendulums of length s and bob mass m swing in a common vertical plane and are attached to two different support points. If the masses are connected by a spring of constant k, use the Lagrangian approach to formulate the equations of motion. Assume small angles of oscillation.

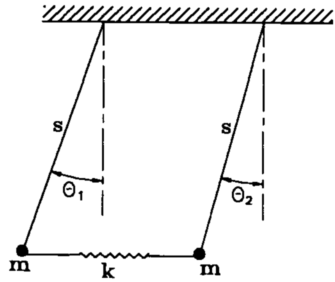

Fig. 4.4 Two simple pendulums.

Solution. θ_1 and θ_2 are the generalized coordinates.

$$T = \tfrac{1}{2}ms^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

$$V = mgs(1 - \cos\theta_1) + mgs(1 - \cos\theta_2) + \tfrac{1}{2}ks^2(\theta_1 - \theta_2)^2$$

$$L = T - V$$

$$L = \tfrac{1}{2}ms^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - mgs(1 - \cos\theta_1) - mgs(1 - \cos\theta_2) - \tfrac{1}{2}ks^2(\theta_1 - \theta_2)^2$$

Working out the derivatives gives

$$\frac{\partial L}{\partial \dot{\theta}_1} = ms^2\dot{\theta}_1, \qquad \frac{\partial L}{\partial \theta_1} = -mgs\sin\theta_1 - ks^2(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = ms^2\dot{\theta}_2, \qquad \frac{\partial L}{\partial \theta_2} = -mgs\sin\theta_2 + ks^2(\theta_1 - \theta_2)$$

Hence the equations of motion are

$$ms^2\ddot{\theta}_1 + mgs\theta_1 + ks^2(\theta_1 - \theta_2) = 0 \tag{4.27}$$

$$ms^2\ddot{\theta}_2 + mgs\theta_2 - ks^2(\theta_1 - \theta_2) = 0 \tag{4.28}$$

Example 4.4

A solid cylinder of radius r and weight w rolls without slipping along a circular path of radius R as shown in Fig. 4.5. Determine the Lagrangian function and the equation of motion.

Solution. From the conditions of no slippage, we have

$$(R - r)\dot{\theta} = r\dot{\phi}$$

The kinetic energy is

$$T = \frac{1}{2}\frac{w}{g}(R - r)^2\dot{\theta}^2 + \frac{1}{2}I_0\dot{\phi}^2$$

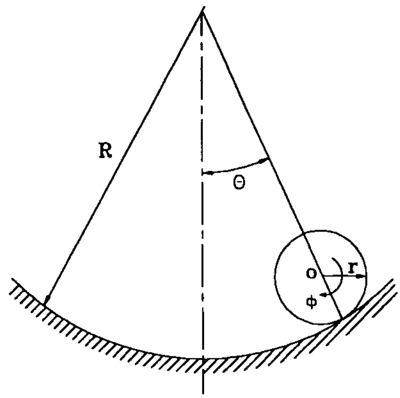

Fig. 4.5 Cylinder rolling on a circular path.

where $I_0 = \frac{1}{2}(w/g)r^2$. Therefore,

$$T = \frac{3}{4}\frac{w}{g}(R-r)^2\dot{\theta}^2$$

$$V = w(R-r)(1-\cos\theta)$$

The Lagrangian function is

$$L = T - V = \frac{3}{4}\frac{w}{g}(R-r)^2\dot{\theta}^2 - w(R-r)(1-\cos\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{3}{2}\frac{w}{g}(R-r)^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -w(R-r)\sin\theta$$

Hence the equation of motion is

$$\frac{3}{2}\frac{w}{g}(R-r)^2\ddot{\theta} + w(R-r)\sin\theta = 0$$

or

$$\ddot{\theta} + \frac{2g}{3(R-r)}\sin\theta = 0 \tag{4.29}$$

Example 4.5

Find the equations of motion for a particle with mass m in three-dimensional space for the following different coordinates: 1) rectangular, 2) cylindrical, and 3) spherical.

Solution. 1) Rectangular coordinates:

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Using the first form of Lagrange's equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \qquad \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial T}{\partial x} = 0, \qquad Q_x = F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x} = F_x$$

Hence we have

$$m\ddot{x} = F_x \tag{4.30}$$

Similarly,

$$m\ddot{y} = F_y \tag{4.31}$$

$$m\ddot{z} = F_z \tag{4.32}$$

2) Cylindrical coordinates:

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z$$

$$\dot{x} = \dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi$$

$$\dot{y} = \dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi$$

$$\dot{z} = \dot{z}$$

Hence,

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$$

For the coordinate ρ, we have

$$\frac{\partial T}{\partial \dot{\rho}} = m\dot{\rho}, \qquad \frac{\partial T}{\partial \rho} = m\rho\dot{\phi}^2$$

$$Q_\rho = F_x\frac{\partial x}{\partial \rho} + F_y\frac{\partial y}{\partial \rho} + F_z\frac{\partial z}{\partial \rho}$$

$$= F_x\cos\phi + F_y\sin\phi = \boldsymbol{F}\cdot\boldsymbol{e}_\rho = F_\rho \tag{4.33}$$

where F_ρ is the component of force along direction ρ. Plugging into Eq. (4.22) gives

$$m\ddot{\rho} - m\rho\dot{\phi}^2 = F_\rho$$

For the coordinate ϕ,

$$\frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi}, \qquad \frac{\partial T}{\partial \phi} = 0$$

$$Q_\phi = F_x\frac{\partial x}{\partial \phi} + F_y\frac{\partial y}{\partial \phi} + F_z\frac{\partial z}{\partial \phi} = -F_x\rho\sin\phi + F_y\rho\cos\phi = \rho\boldsymbol{F}\cdot\boldsymbol{e}_\phi = \rho F_\phi$$

where $\boldsymbol{e}_\phi = -\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j}$ as given in Eq. (2.8). Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\rho^2\dot{\phi}) = \rho F_\phi \tag{4.34}$$

For the coordinate z, the equation is the same as in the rectangular coordinates

$$m\ddot{z} = F_z \tag{4.35}$$

3) Spherical coordinates: In Chapter 2, the relationship between spherical coordinates and rectangular coordinates was already introduced. From Eqs. (2.11) and (2.12), we have

$$\boldsymbol{r} = r\boldsymbol{e}_r$$

$$x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = r\,\sin\theta\,\cos\phi\boldsymbol{i} + r\,\sin\theta\,\sin\phi\boldsymbol{j} + r\,\cos\theta\,\boldsymbol{k}$$

That is,

$$x = r\,\sin\theta\,\cos\phi$$

$$y = r\,\sin\theta\,\sin\phi$$

$$z = r\,\cos\theta$$

We also have, from Eq. (2.15),

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + r\dot{\phi}\sin\theta\boldsymbol{e}_\phi$$

Hence,

$$T = \tfrac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + (r\dot{\phi}\sin\theta)^2]$$

For the coordinate r,

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r}, \qquad \frac{\partial T}{\partial r} = m(r\dot{\theta}^2 + r\dot{\phi}^2\sin^2\theta)$$

$$Q_r = F_x\frac{\partial x}{\partial r} + F_y\frac{\partial y}{\partial r} + F_z\frac{\partial z}{\partial r} = F_x\sin\theta\,\cos\phi$$
$$+\; F_y\sin\theta\,\sin\phi + F_z\cos\theta = \boldsymbol{F}\cdot\boldsymbol{e}_r = F_r$$

With the use of Eq. (4.22), we obtain

$$m(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta) = F_r \tag{4.36}$$

for the equation of motion in the radial direction. For the coordinate θ,

$$\frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}, \qquad \frac{\partial T}{\partial \theta} = mr^2\dot{\phi}^2\sin\theta\,\cos\theta$$

$$Q_\theta = F_x r\,\cos\theta\,\cos\phi + F_y r\,\cos\theta\,\sin\phi - F_z r\,\sin\theta = r\boldsymbol{F}\cdot\boldsymbol{e}_\theta = rF_\theta$$

Hence the equation of motion in the direction of θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) - mr^2\dot{\phi}^2 \sin\theta\cos\theta = rF_\theta \tag{4.37}$$

Note that the generalized force in θ direction is a torque. Similarly, for the coordinate ϕ

$$\frac{\partial T}{\partial \dot{\phi}} = mr^2 \sin^2\theta\dot{\phi}, \qquad \frac{\partial T}{\partial \phi} = 0$$

$$\begin{aligned} Q_\phi &= -F_x r \sin\theta \sin\phi + F_y r \sin\theta \cos\phi \\ &= r \sin\theta \boldsymbol{F} \cdot \boldsymbol{e}_\phi = r \sin\theta F_\phi \end{aligned}$$

The equation of motion in the direction of ϕ is, therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2 \sin^2\theta\dot{\phi}) = r \sin\theta F_\phi \tag{4.38}$$

Equations (4.37) and (4.38) can be simplified to

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2 \sin\theta \cos\theta) = F_\theta \tag{4.39}$$

$$m(2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\ddot{\phi}) = F_\phi \tag{4.40}$$

Note that the acceleration terms on the left sides of Eqs. (4.36), (4.39), and (4.40) agree well with the expression in Eq. (2.16).

Example 4.6

Suppose that a person of mass M playing on a swing is modeled as a point mass $(M - m)$ at the end of the rope and a small mass m moving around $M - m$ at radius a and angular speed of ω as shown in Fig. 4.6. Find the equation of motion for this system.

Solution. Velocity of $(M - m)$

$$V_{M-m} = s\dot{\theta}(\cos\theta i + \sin\theta j) \tag{4.41}$$

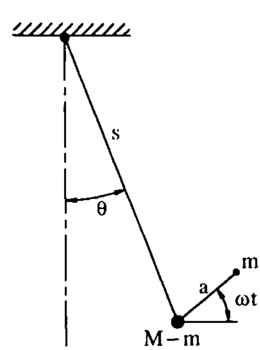

Fig. 4.6 Person playing on a swing.

Velocity of m is

$$V_m = s\dot{\theta}(\cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}) - a\omega\sin\omega t\,\boldsymbol{i} + a\omega\cos\omega t\,\boldsymbol{j} \tag{4.42}$$

The kinetic energy of the system is

$$T = \tfrac{1}{2}(M - m)V_{M-m}^2 + \tfrac{1}{2}mV_m^2$$

$$T = \tfrac{1}{2}(M - m)(s\dot{\theta})^2 + \tfrac{1}{2}m[(s\dot{\theta}\cos\theta - a\omega\sin\omega t)^2 + (s\dot{\theta}\sin\theta + a\omega\cos\omega t)^2] \tag{4.43}$$

The potential energy is

$$\begin{aligned} V &= (M - m)gs(1 - \cos\theta) + mg[s(1 - \cos\theta) + a\sin\omega t] \\ &= Mgs(1 - \cos\theta) + mga\sin\omega t \end{aligned} \tag{4.44}$$

The Lagrangian function for the system is

$$\begin{aligned} L = T - V &= \tfrac{1}{2}M(s\dot{\theta})^2 + \tfrac{1}{2}m[(a\omega)^2 - 2(a\omega\sin\omega t)(s\dot{\theta}\cos\theta) \\ &\quad + 2(a\omega\cos\omega t)(s\dot{\theta}\sin\theta)] - Mgs(1 - \cos\theta) - mga\sin\omega t \\ &= \tfrac{1}{2}M(s\dot{\theta})^2 + \tfrac{1}{2}m[(a\omega)^2 + 2as\omega\dot{\theta}\sin(\theta - \omega t)] \\ &\quad - Mgs(1 - \cos\theta) - mga\sin\omega t \end{aligned} \tag{4.45}$$

To find the equation of motion, we obtain

$$\frac{\partial L}{\partial \dot{\theta}} = Ms^2\dot{\theta} + mas\omega\sin(\theta - \omega t)$$

$$\frac{\partial L}{\partial \theta} = mas\omega\,\dot{\theta}\cos(\theta - \omega t) - Mgs\sin\theta$$

Substituting the preceding equations in Eq. (4.22) leads us to

$$Ms^2\ddot{\theta} + mas\omega\cos(\theta - \omega t)(\dot{\theta} - \omega) - mas\omega\,\dot{\theta}\cos(\theta - \omega t) + Mgs\sin\theta = 0$$

Rearranging, we obtain the equation of motion as

$$\ddot{\theta} + \frac{g}{s}\sin\theta = \frac{ma}{Ms}\omega^2\cos(\omega t - \theta) \tag{4.46}$$

Note that the term on the right-hand side is the force causing the swing to oscillate to a large angle. Resonance can take place as

$$\omega = \sqrt{g/s}$$

Through these examples it is easily seen that using the Lagrangian equation for deriving equations of motion for conservative systems is very simple and systematic. All we need are the expressions for potential and kinetic energy.

4.3 Hamilton's Principle

Hamilton's principle is an approach parallel to the Lagrangian equations. From here readers can get a deeper feeling about equations describing a dynamic system. Similar to Lagrange's approach, the Hamiltonian function H is defined as

$$H \equiv \sum_j \dot{q}_j p_j - L = H(p, q, t) \tag{4.47}$$

where $p_j =$ the generalized momenta $= \partial L/\partial \dot{q}_j$. Taking the total derivative of Eq. (4.47) gives us

$$\begin{aligned} \mathrm{d}H &= \sum_j \dot{q}_j \mathrm{d}p_j + \sum_j p_j \mathrm{d}\dot{q}_j - \left[\sum_j \frac{\partial L}{\partial q_j} \mathrm{d}q_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \mathrm{d}\dot{q}_j + \frac{\partial L}{\partial t} \mathrm{d}t \right] \\ &= \sum_j \dot{q}_j \mathrm{d}p_j - \sum_j \frac{\partial L}{\partial q_j} \mathrm{d}q_j - \frac{\partial L}{\partial t} \mathrm{d}t \end{aligned} \tag{4.48}$$

Also, we have

$$\mathrm{d}H = \sum_j \frac{\partial H}{\partial p_j} \mathrm{d}p_j + \sum_j \frac{\partial H}{\partial q_j} \mathrm{d}q_j + \frac{\partial H}{\partial t} \mathrm{d}t \tag{4.49}$$

Compare Eq. (4.48) to Eq. (4.49), we obtain

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \tag{4.50a}$$

$$-\frac{\partial H}{\partial q_j} = \frac{\partial L}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \dot{p}_j \tag{4.50b}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{4.50c}$$

Equations (4.50a) and (4.50b) are called Hamilton's canonical equations for a conservative system because, in the intermediate step of deriving Eq. (4.50b), the conservative condition is used. Furthermore, for a conservative system

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t} &= \sum_j \left(\frac{\partial H}{\partial q_j} \dot{q}_j + \frac{\partial H}{\partial p_j} \dot{p}_j \right) + \frac{\partial H}{\partial t} \\ &= \sum_j (-\dot{p}_j \dot{q}_j + \dot{q}_j \dot{p}_j) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \end{aligned} \tag{4.51}$$

To interpret the meaning of H, let us consider a case that happens often in dynamics; the position vectors are functions of q only:

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q) \quad i = 1, 2, \ldots, N$$

Then

$$v_i = \frac{d\boldsymbol{r}_i}{dt} = \sum_k \frac{\partial \boldsymbol{r}_i}{\partial q_k}\dot{q}_k$$

$$T = \frac{1}{2}\sum_i m_i v_i \cdot v_i = \frac{1}{2}\sum_i m_i \sum_{k,l}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_l}\right)\dot{q}_k\dot{q}_l$$

and

$$\sum_j \dot{q}_j p_j = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j}$$

$$= \sum_j \dot{q}_j \left[\frac{1}{2}\sum_i m_i \sum_{k,l}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_l}\right)\frac{\partial}{\partial \dot{q}_j}\dot{q}_k\dot{q}_l\right]$$

Looking into the details of the partial derivative in the last expression, we find

$$\sum_{k,l}\frac{\partial}{\partial \dot{q}_j}(\dot{q}_k\dot{q}_l) = \sum_{k,l}(\dot{q}_l\delta_{j,k} + \dot{q}_k\delta_{j,l}) = 2\dot{q}_j$$

Therefore,

$$\sum_j \dot{q}_j p_j = \sum_j \dot{q}_j \left[\frac{1}{2}\sum_i m_i \frac{\partial \boldsymbol{r}_i}{\partial q_j}\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_j}(2\dot{q}_j)\right]$$

$$= \sum_j \left[\sum_i m_i \left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j\right)^2\right] = 2T \tag{4.52}$$

Substituting Eq. (4.52) into Eq. (4.47), we find

$$H = 2T - L = 2T - (T - V) = T + V \tag{4.53}$$

Therefore H is the total energy of the system if the various $\boldsymbol{r}_i$ are functions of q only. For a conservative system

$$T + V = \text{const}$$

$$H = \text{const}$$

That means

$$\frac{dH}{dt} = 0 \qquad \text{and} \qquad \frac{\partial H}{\partial t} = 0 \tag{4.54}$$

For a nonconservative system, the Lagrangian equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \mathcal{F}_j \tag{4.55}$$

With this general expression, Eq. (4.50b) becomes

$$-\frac{\partial H}{\partial q_j} = \frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \mathcal{F}_j = \dot{p}_j - \mathcal{F}_j \tag{4.56}$$

Equation (4.51) becomes

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_j [(-\dot{p}_j + \mathcal{F}_j)\dot{q}_j + \dot{q}_j \dot{p}_j] + \frac{\partial H}{\partial t} = \sum_j \mathcal{F}_j \dot{q}_j + \frac{\partial H}{\partial t} \tag{4.57}$$

Therefore, Hamilton's canonical equations are true only for conservative systems. In general the total derivative of H with respect to time is not the partial derivative of H with respect to time.

Example 4.7

Consider a spherical pendulum consisting of a point mass m that moves under gravity on a smooth spherical surface with radius a. The gravitational force is along the downward vertical. In terms of spherical angles θ and ϕ as shown in Fig. 2.2, except that θ is the angle between the position vector of mass m and the downward vertical axis, the kinetic and potential energies are

$$T = \tfrac{1}{2} m a^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

$$V = -mga\cos\theta$$

Find the equations of motion for the mass m 1) from Lagrange's equation and 2) from Hamilton's principle.

Solution. 1) Lagrange's equation:

$$L = T - V = \tfrac{1}{2} m a^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + mga\cos\theta$$

For the coordinate θ,

$$\frac{\partial L}{\partial \dot{\theta}} = m a^2 \dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = m a^2 \dot{\phi}^2 \sin\theta\cos\theta - mga\sin\theta$$

Substituting the preceding expressions into Eq. (4.23), we find

$$m a^2 \ddot{\theta} - m a^2 \dot{\phi}^2 \sin\theta\cos\theta + mga\sin\theta = 0 \tag{4.58}$$

For the coordinate ϕ,

$$\frac{\partial L}{\partial \dot{\phi}} = m a^2 \dot{\phi}\sin^2\theta$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m a^2 \dot{\phi}\sin^2\theta) = 0$$

$$\dot{\phi}\sin^2\theta = \text{const} \tag{4.59}$$

2) Hamilton's principle: In spherical coordinates

$$\boldsymbol{r} = r(\sin\theta\cos\phi\boldsymbol{i} + \sin\theta\sin\phi\boldsymbol{j} + \cos\theta\boldsymbol{k}) = \boldsymbol{r}(r, \theta, \phi)$$

Hence,

$$H = T + V = \tfrac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mga\cos\theta$$

In Hamilton's principle, however, H is to be expressed in terms of generalized coordinates q, generalized momenta p, and time t:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} \tag{4.60}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ma^2\dot{\phi}\sin^2\theta \tag{4.61}$$

With the use of Eqs. (4.60) and (4.61), we have

$$H = \frac{1}{2}\frac{p_\theta^2}{ma^2} + \frac{1}{2}\frac{p_\phi^2}{ma^2\sin^2\theta} - mga\cos\theta \tag{4.62}$$

Taking the partial derivatives of H with respect to θ and ϕ, we have

$$\frac{\partial H}{\partial \theta} = -\frac{p_\phi^2}{ma^2\sin^3\theta}\cos\theta + mga\sin\theta$$

$$\frac{\partial H}{\partial \phi} = 0$$

Rewrite the canonical equations

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta, \qquad \frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

With the help of Eqs. (4.60) and (4.61), and the canonical equations, we find

$$-\frac{p_\phi^2}{ma^2\sin^4\theta}\sin\theta\cos\theta + mga\sin\theta = -ma^2\ddot{\theta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\phi}\sin^2\theta) = 0$$

Further simplifying the preceding equation, we obtain

$$-ma^2\dot{\phi}^2\sin\theta\cos\theta + mga\sin\theta = -ma^2\ddot{\theta} \tag{4.63}$$

$$\dot{\phi}\sin^2\theta = \text{const} \tag{4.64}$$

Equations (4.63) and (4.64) are the same as Eqs. (4.58) and (4.59) obtained in part 1.

4.4 Lagrangian Equations with Constraints

In general there are two types of constraints in dynamics: holonomic and nonholonomic. When the relationship between generalized coordinates can be written as

$$f_i(q_1, q_2, \ldots, q_n, t) = 0 \quad i = 1, 2, \ldots, m \tag{4.65}$$

where $m < n$, the constraints of this form are known as holonomic constraints. Because of these m constraint equations, the various nq_j are not independent. In principle, there are only $(n - m)$ independent generalized coordinates, and $(n - m)$ Lagrangian equations for solving these q_i as functions of time. The remaining q_i can be obtained through Eqs. (4.65) already given.

Many problems, however, may be formulated differently such that the generalized coordinates can be reduced at the beginning. For example, let us consider the case of a double pendulum (Fig. 4.7). The two point masses m_1 and m_2 can be specified by (x_1, y_1) and (x_2, y_2) in the plane containing the double pendulum. The rods of length L_1 and L_2 are considered to be rigid and massless. The constraint equations are of the form

$$x_1^2 + y_1^2 = L_1^2$$

$$(x_2 - x_1)^1 + (y_2 - y_1)^2 = L_2^2$$

Because of these, we simply choose θ_1 and θ_2 as generalized coordinates and the equations of motion are simplified. On the other hand, when the constraint equations are written in the form

$$\sum_{j=1}^{n} C_{kj}\,\mathrm{d}q_j + C_{kt}\,\mathrm{d}t = 0 \quad k = 1, 2, \ldots, m \tag{4.66}$$

where the various C are, in general, functions of the generalized coordinates and time. Constraints of this form are known as nonholonomic constraints. While deriving the Lagrangian equation of the first form, there is a step written in

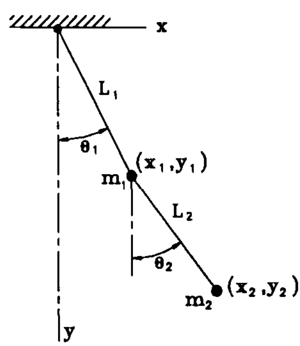

Fig. 4.7 Double pendulum.

Eq. (4.21) as

$$\sum_j \left[Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j} \right] \delta q_j = 0$$

At that moment, because q_j is independent throughout, the terms in the brackets were set to zero. Now q_j is not independent and cannot be set to zero. The general expression for the generalized force, however, is still valid, i.e.,

$$Q_j = -\frac{\partial V}{\partial q_j} + \mathcal{F}_j$$

Furthermore, to broaden our considerations, the nonconservative forces may be treated as a combination of constraint force $\mathcal{F}_{cj}$ and the other nonconservative force $\mathcal{F}_{oj}$. Substituting this expression into Eq. (4.21), we have

$$\sum_j \left[\frac{\partial L}{\partial q_j} + \mathcal{F}_{cj} + \mathcal{F}_{oj} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \right] \delta q_j = 0 \tag{4.67}$$

Let Eq. (4.66) be multiplied by λ_k and summed over k throughout. Adding that to Eq. (4.67) gives

$$\sum_{j=1}^{n} \left[\frac{\partial L}{\partial q_j} + \mathcal{F}_{cj} + \mathcal{F}_{oj} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_k \lambda_k C_{kj} \right] \delta q_j + \sum_{k=1}^{m} \lambda_k C_{kt}\,\mathrm{d}t = 0$$

Rearranging the equation leads to

$$\sum_{j=1}^{n} \left[\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \mathcal{F}_{oj} + \sum_k \lambda_k C_{kj} \right] \mathrm{d}q_j$$
$$+ \sum_{j=1}^{n} \mathcal{F}_{cj}\,\mathrm{d}q_j + \sum_{k=1}^{m} \lambda_k C_{kt}\,\mathrm{d}t = 0$$

The preceding equation can be considered a combination of two equations, which is proved here. The two equations are

$$\sum_{j=1}^{n} \left[\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \mathcal{F}_{oj} + \sum_k \lambda_k C_{kj} \right] \mathrm{d}q_j = 0 \tag{4.68}$$

and

$$\sum_{j=1}^{n} \mathcal{F}_{cj}\,\mathrm{d}q_j + \sum_{k=1}^{m} \lambda_k C_{kj}\,\mathrm{d}t = 0 \tag{4.69}$$

In Eq. (4.68), note that only $(n - m)$ q_j is independent, but there are m arbitrary λ_k values. Choose m λ_k values such that the sum of four terms in the bracket is zero for m brackets. These various $m q_j$ are presumed to be dependent coordinates. Then the remaining q_j are independent, and the sum of the four terms in the bracket are always zero, i.e.,

$$\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \mathcal{F}_{oj} + \sum_k \lambda_k C_{kj} = 0 \quad j = 1, 2, \ldots, n \tag{4.70}$$

Now let us consider Eq. (4.69). When Eq. (4.66) is multiplied by λ_k and summed over k throughout, we have

$$\sum_k \lambda_k c_{kt}\,\mathrm{d}t = -\sum_k \sum_j \lambda_k C_{kj}\,\mathrm{d}q_j \tag{4.71}$$

Substitute this into Eq. (4.69), we find that

$$\sum_j \mathcal{F}_{cj}\,\mathrm{d}q_j - \sum_k \sum_j \lambda_k C_{kj}\,\mathrm{d}q_j = 0$$

or

$$\sum_j \left[\mathcal{F}_{cj} - \sum_k \lambda_k C_{kj}\right] \mathrm{d}q_j = 0 \tag{4.72}$$

But,

$$\mathcal{F}_{cj} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} - \mathcal{F}_{oj}$$

With the use of this equation for the nonconservative force in Eq. (4.72), we obtain

$$\sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q_j}\right) - \frac{\partial L}{\partial q_j} - \mathcal{F}_{oj} - \sum_k \lambda_k C_{kj}\right] \mathrm{d}q_j = 0 \tag{4.73}$$

Equation (4.73) multiplied by (-1) is identical to Eq. (4.68), which has been proved to be true. Therefore, Eq. (4.69) is also true. Summarizing all the equations, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \sum_k \lambda_k C_{kj} + \mathcal{F}_{oj} \quad j = 1, 2, \ldots, n \tag{4.74}$$

$$\mathcal{F}_{cj} = \sum_k \lambda_k c_{kj} \qquad j = 1, 2, \ldots, n \tag{4.75}$$

$$\sum_j c_{kj}\,\mathrm{d}q_j + c_{kt}\,\mathrm{d}t = 0 \qquad k = 1, 2, \ldots, m \tag{4.76}$$

Totally, there are $2n + m$ equations for determining $n q_j$, $n \mathcal{F}_{cj}$ and $m\lambda_k$; λ_k is called the Lagrange multiplier, $\mathcal{F}_{cj}$ represents constraint forces, and $\mathcal{F}_{oj}$, the other nonconservative forces.

Example 4.8

A four-wheel wagon is modeled as a mass m in translational motion and four wheels in rotational motion (see Fig. 4.8). The mass m includes the four wheels. The moment of inertia for the four wheels with respect to the rotating axes is I. Determine the required coefficient of friction between tires and the pavement for the wagon to move without slipping down the slope inclined at angle ϕ.

Solution. Kinetic energy:

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2$$

Potential energy:

$$V = mgx\sin\phi$$

Constraint equation:

$$\mathrm{d}x - r\mathrm{d}\theta = 0$$

where r is the radius of wheels. The Lagrangian function is

$$L = T - V = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2 - mgx\sin\phi$$

For the x coordinate,

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial x} = -mg\sin\phi$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) + mg\sin\phi = \lambda = \mathcal{F}_x$$

or

$$m\ddot{x} = \mathcal{F}_x - mg\sin\phi \tag{4.77}$$

For the θ coordinate,

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}, \qquad \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(I\dot{\theta}) = -\lambda r$$

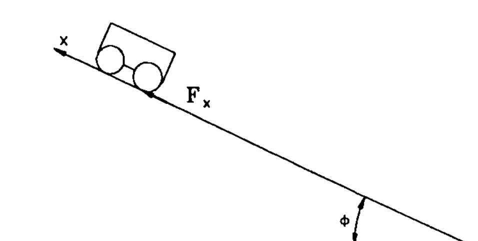

Fig. 4.8 Wagon rolling down inclined plane.

or

$$I\ddot{\theta} = -\lambda r = -\mathcal{F}_x r \tag{4.78}$$

From the constraint equation we have

$$\dot{x} = r\dot{\theta}$$

$$\ddot{x} = r\ddot{\theta}$$

Combining the preceding equation with Eqs. (4.77) and (4.78), we find

$$\mathcal{F}_x = \frac{g}{(1/m + r^2/I)} \sin\phi$$

Because $\mathcal{F}_x = \mu(mg\cos\phi)$

$$\mu = \frac{I}{I + mr^2} \tan\phi \tag{4.79}$$

where μ is the required frictional coefficient.

Example 4.9

Suppose that a car is just started and is to be driven without slipping on horizontal ground covered with ice. With the use of Lagrangian equations that are constrained, find the equations to describe the motion and find the required frictional coefficient between the tires and the ice. Explain why the driver should not attempt to accelerate rapidly. Assume that the mass of the car is M, the moment of inertia of wheels is I, and the torque exerted on the wheels is T_r. The weight of the car is distributed evenly on all four wheels, and this is a four-wheel-drive vehicle.

Solution. Kinetic energy:

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2$$

Potential energy:

$$V = 0$$

Constraint equation:

$$\mathrm{d}x - r\,\mathrm{d}\theta = 0$$

The nonconservative generalized force in the θ direction is T_r, and the Lagrangian function is

$$L = T - V = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2$$

For the x coordinate,

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x}, \qquad \frac{\partial L}{\partial x} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(M\dot{x}) = \lambda = \mathcal{F}_x$$

This is the equation of motion in the x direction. For the θ coordinate,

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}, \qquad \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(I\dot{\theta}) = T_r - \lambda r = T_r - \mathcal{F}_x r$$

This is the equation of motion in the θ direction. From the constraint equation, we have

$$\ddot{x} = r\ddot{\theta}$$

Combining the equations of motion together with the preceding equation, we obtain

$$I\ddot{\theta} = I\frac{\ddot{x}}{r} = \frac{I}{r}\frac{\mathcal{F}_x}{M} = T_r - \mathcal{F}_x r$$

Rearranging, we find

$$\mathcal{F}_x\left(\frac{I}{Mr} + r\right) = T_r$$

Because friction can be expressed as the product of the frictional coefficient and its weight, the frictional coeffcient is determined as

$$\mu = \frac{T_r}{(Ig/r + Mgr)}$$

where g is gravitational acceleration. Hence the required frictional coefficient is higher as torque increases. The driver should not try to accelerate rapidly, because, as the torque increases, the required frictional coefficient to avoid spinning wheels on ice will exceed the actual frictional coefficient.

Example 4.10

Consider a block of mass m sliding on a straight rod without friction as a case for a time-dependent constraint. The rod is rotating in the x–y plane that is perpendicular to the gravitational force. The rod is rotating at constant velocity ω. Find 1) the radial position of the block as a function of time and 2) the constraint force from the rod on the block. A similar problem has been presented in Example 2.3. The physical conditions are shown in Fig. 2.7.

Solution. The r and θ are the generalized coordinates. The constraint equation is

$$\theta = \omega t$$

or

$$\mathrm{d}\theta - \omega\,\mathrm{d}t = 0 \tag{4.80}$$

so that

$$C_r = 0, \qquad C_\theta = 1, \qquad C_t = -\omega$$

The kinetic energy is

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

The potential energy is a constant that is set to zero, i.e.,

$$V = 0$$

Therefore,

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

1) For the equation in the r direction,

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) - mr\omega^2 = 0$$

$$\ddot{r} - \omega^2 r = 0$$

$$\begin{aligned} r &= A\cosh\omega t + B\sinh\omega t \\ &= r_0\cosh\omega t + (\dot{r}_0/\omega)\sinh\omega t \end{aligned} \tag{4.81}$$

where r_0 and $\dot{r}_0$ are the initial position and velocity of the block along the r direction.

2) For the constraint force,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 2m\omega r\dot{r} = \lambda \tag{4.82}$$

$$\mathcal{F}_\theta = 2m\omega r\dot{r}$$

Here, the generalized constraint force is a torque. The force between the rod and the block is $2m\omega\dot{r}$.

4.5 Calculus of Variations

The calculus of variations is a totally different approach from Lagrangian equations. It is a method for us to determine conditions under which the integral of a given function will reach a maximum or minimum. But it can also reach Lagrange's equation for a conservative system. Because of that it is included in this chapter.

To understand the method, let us consider a function f that is to be integrated over a path $y(x)$. The starting point of the path is (x_1, y_1), and the end point is (x_2, y_2) as shown in Fig. 4.9. Assume that the function f can be written as

$$f = f(y, y', x)$$

where y and y' and x are independent variables, although y and y' are functions

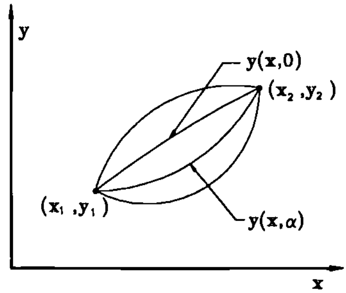

Fig. 4.9 Paths for line integration.

of x. The integral of f is then

$$I = \int_{x_1}^{x_2} f(y, y', x)\mathrm{d}x \tag{4.83}$$

Clearly, the result of the integral depends on the path $y(x)$ chosen. Here we want to determine a particular path $y(x)$, so that it makes the integral to reach the extremum. To reach that goal, we let

$$y(x, \alpha) = y(x, 0) + \alpha g(x) \tag{4.84}$$

where $g(x) = \partial y/\partial\alpha$ and $g(x_1) = g(x_2) = 0$. This means that the path is varied from $y(x)$ to $y(x, \alpha)$. The condition for the extremum of the integral is then

$$\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} = 0 \tag{4.85}$$

From Eq. (4.83), we have

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right)\mathrm{d}x \tag{4.86}$$

In the preceding equation, the second term on the right can be simplified with the use of integration by parts, i.e.,

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial x \partial \alpha}\mathrm{d}x = \frac{\partial f}{\partial y'}\frac{\partial y}{\partial \alpha}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha}\mathrm{d}x = \frac{\partial f}{\partial y'}\bigg|_{x=x_2} g(x_2)$$
$$- \frac{\partial f}{\partial y'}\bigg|_{x=x_1} g(x_1) - \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha}\mathrm{d}x = -\int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha}\mathrm{d}x$$

Substituting the preceding equation into Eq. (4.86), we obtain

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right]\frac{\partial y}{\partial \alpha}\mathrm{d}x$$

Now multiplying the equation by $\mathrm{d}\alpha$ and setting α to 0 and writing

$$\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} \mathrm{d}\alpha = \delta I$$

$$\left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=0} \mathrm{d}\alpha = \delta y$$

we find

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right] \delta y \,\mathrm{d}x = 0$$

Because δy is arbitrary and not zero as $x_1 < x < x_2$, the terms in the brackets must be zero, i.e.,

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0 \tag{4.87}$$

This equation is known as the Euler–Lagrange equation. Note that if we change symbols, $f \to L$, $y' \to \dot{q}$, $y \to q$, and $x \to t$, we can write Eq. (4.87) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

which is Lagrange's equation for a conservative system. Equation (4.87) is the tool for us to find $y(x)$ for I to become the extremum. It is similar to Lagrange's equation, from which we find $q(t)$. For a special case, when f is not an explicit function of x, Eq. (4.87) can be further simplified. Multiplying Eq. (4.87) by y', we have

$$y'\frac{\partial f}{\partial y} - y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Adding and subtracting $(\partial f/\partial y')y''$ and also adding $\partial f/\partial x$, which is zero anyway, we obtain

$$\frac{\partial f}{\partial y'}y'' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y'}y'' - y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Rewrite the first three terms as $\mathrm{d}f/\mathrm{d}x$ and the last two terms as $(\mathrm{d}/\mathrm{d}x)[y'(\partial f/\partial y')]$; we find

$$\frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'}\right) = 0$$

or

$$f - y'\frac{\partial f}{\partial y'} = \text{const} \tag{4.88}$$

which is even simpler than Eq. (4.87) for finding $y(x)$. It will become clear after studying a few examples later.

On the other hand, sometimes we like to have

$$I = \int_{x_1}^{x_2} f(y, y', x)\,\mathrm{d}x \tag{4.89}$$

to reach the extremum, but notice a condition is imposed, such as,

$$\int_{x_1}^{x_2} \sigma(y, y', x)\,\mathrm{d}x = C_0 \tag{4.90}$$

To treat this type of problem, we multiply Eq. (4.90) by λ and add that to Eq. (4.89), then we have

$$I + \lambda c_0 = I' = \int_{x_1}^{x_2} (f + \lambda\sigma)\,\mathrm{d}x$$

$$= \int_{x_1}^{x_2} F(y, y', x)\,\mathrm{d}x$$

$$\delta I' = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right] \delta y\,\mathrm{d}x = 0$$

in which $F(y, y', x) = f + \lambda\sigma$. Similar to the way we find Eq. (4.87), we obtain

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{4.91}$$

From this equation, $y(x, \lambda)$ will be found. The constant λ then is determined by Eq. (4.90), which is equivalent to the constraint equation already discussed.

Example 4.11

A geodesic on a given surface is a curve, lying on that surface, along which the distance between two points is shortest. Determine the equation of geodesic on a right circular cylinder.

Solution. The radius of the cylinder is a. Take the z axis along the axis of the cylinder. The two points on the cylindrical surface are (z_1, θ_1) and (z_2, θ_2). The distance between two points is

$$S = \int_{\theta_1}^{\theta_2} \sqrt{a^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

Therefore,

$$f(z, z', \theta) = \sqrt{a^2 + z'^2}$$

$$\frac{\partial f}{\partial z} = 0, \qquad \frac{\partial f}{\partial \theta} = 0$$

$$\frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{a^2 + z'^2}}$$

Using Eq. (4.88), we have

$$f = \sqrt{a^2 + z'^2} = \text{const}$$

or

$$z' = \frac{\mathrm{d}z}{\mathrm{d}\theta} = \text{const}$$

$$z = c_0\theta + c_1$$

Therefore, the equation for the geodesic on a circular cylinder is found to be

$$z = \frac{z_2 - z_1}{\theta_2 - \theta_1}\theta + \frac{z_1\theta_2 - z_2\theta_1}{\theta_2 - \theta_1} \tag{4.92}$$

Example 4.12

Just to illustrate the point that the calculus of variations also leads to the Lagrangian equation for a conservative system, let us consider a particle of mass m freely falling under gravity. Find the equation of motion by considering

$$I = \int_{t_1}^{t_2} L(y, \dot{y}, t)\mathrm{d}t$$

Solution. The energies of the system are

$$T = \tfrac{1}{2}m\dot{y}^2, \qquad V = mg(y - y_0)$$

$$L = T - V = \tfrac{1}{2}m\dot{y}^2 - mg(y - y_0)$$

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 = -mg - m\ddot{y}$$

The equation of motion is

$$\ddot{y} = -g$$

Note that the y axis is taken vertically upward.

Example 4.13

The surface area for a body revolving with the x axis can be expressed as

$$I = 2\pi \int_{x_1}^{x_2} y(1 + y'^2)^{\frac{1}{2}}\,\mathrm{d}x$$

Determine the function $y(x)$ that minimizes the integral I.

Solution. Rewrite the integral as

$$\frac{I}{2\pi} = \int_{x_1}^{x_2} y(1 + y'^2)^{\frac{1}{2}}\,\mathrm{d}x$$

Here the function f is

$$f(y, y', x) = y(1 + y'^2)^{\frac{1}{2}}$$

which is not an explicit function of x. Using Eq. (4.88), we find

$$y(1 + y'^2)^{\frac{1}{2}} - y' \frac{yy'}{(1 + y'^2)^{\frac{1}{2}}} = c_1$$

or

$$y(1 + y'^2) - yy'^2 = c_1(1 + y'^2)^{\frac{1}{2}}$$

Simplifying leads to

$$y = c_1(1 + y'^2)^{\frac{1}{2}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{y^2}{c_1^2} - 1\right)^{\frac{1}{2}}$$

Integrating yields

$$y = c_1 \cosh\left(\frac{x}{c_1} + c_2\right)$$

where c_1, c_2 are integral constant and can be determined if the two end points are specified.

Example 4.14

Determine the equation for the shortest arc that passes through the points (0, 0) and (1, 0) and encloses a prescribed area A with the x axis (Fig. 4.10).

Solution. According to the given conditions, we have

$$I = \int_0^1 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \tag{4.93}$$

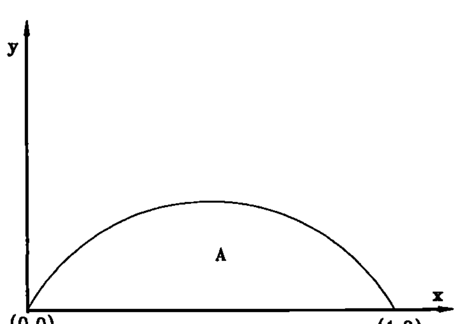

Fig. 4.10 Shortest arc between (0, 0) and (1, 0) but enclosing A.

and

$$A = \int_0^1 y \,\mathrm{d}x \tag{4.94}$$

Hence,

$$\begin{aligned} f &= \sqrt{1+y^2}, \qquad \sigma = y \\ F &= f + \lambda\sigma = \sqrt{1+y^2} + \lambda y \\ \frac{\partial F}{\partial y'} &= \frac{y'}{\sqrt{1+y'^2}}, \qquad \frac{\partial F}{\partial y} = \lambda \end{aligned} \tag{4.95}$$

Using Eq. (4.91), we have

$$\lambda - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0 \tag{4.96}$$

Integrating leads to

$$\frac{y'}{\sqrt{1+y'^2}} = \lambda x + c_1$$

$$y' = \pm \frac{\lambda x + c_1}{\sqrt{1-(\lambda x + c_1)^2}}$$

Integrating again, we find

$$y = \mp \frac{1}{\lambda}\sqrt{1-(\lambda x + c_1)^2} + c_2 \tag{4.97}$$

Applying the boundary conditions (0, 0) and (1, 0), we find

$$c_1 = -\frac{\lambda}{2}, \qquad c_2 = -\frac{1}{\lambda}\sqrt{1-\frac{\lambda^2}{4}} \tag{4.98}$$

Substituting c_1 and c_2 into Eq. (4.97) and using Eq. (4.94), we obtain

$$\begin{aligned} A &= \int_0^1 y \,\mathrm{d}x = \int_0^1 \frac{1}{\lambda}\sqrt{1-\left(\lambda x - \frac{\lambda}{2}\right)^2}\,\mathrm{d}x + c_2 x\Big|_0^1 \\ &= \frac{1}{\lambda^2}\left[\frac{\lambda}{2}\sqrt{1-\frac{\lambda^2}{4}} + \sin^{-1}\frac{\lambda}{2}\right] + c_2 \\ \frac{\lambda}{2} &= \sin\left[\lambda^2 A + \frac{\lambda}{2}\sqrt{1-\frac{\lambda^2}{4}}\right] \end{aligned} \tag{4.99}$$

The value of λ is determined by this equation. And $y(x)$ is written as

$$y = \frac{1}{\lambda}\sqrt{1 - \left(\lambda x - \frac{\lambda}{2}\right)^2} - \frac{1}{\lambda}\sqrt{1 - \frac{\lambda^2}{4}} \qquad (4.100)$$

which can be rewritten in a familiar form as

$$\left(x - \frac{1}{2}\right)^2 + (y - c_2)^2 = \frac{1}{\lambda^2}$$

Therefore the curve is a circular arc with the center at $(\frac{1}{2}, c_2)$ and a radius of $1/\lambda$.

Problems

4.1. Derive the equations of motion for Example 2.1 with the use of Lagrangian equations.

4.2. Derive the equations of motion for Example 2.2 with the use of Lagrangian equations. Make some necessary assumptions to simplify the problem.

4.3. Develop the Lagrangian equation for the momentum equation of the incompressible fluid flow in fluid mechanics.

4.4. Use the result of Problem 4.3 to find the momentum equations in cylindrical and spherical coordinates for incompressible fluid flow in fluid mechanics.

4.5. Suppose a point mass m is attached to one end of a horizontal spring with spring constant k, the other end of which is fixed on a cart that is being moved uniformly in a horizontal plane by an external device with speed v_0. If we take as a generalized coordinate the position x of the mass particle in the stationary system, find the equation of the motion for m, from the following:
(a) The Lagrangian equation.
(b) Hamilton's canonical equations.

4.6. A heavy particle is placed at the top of a vertical hoop. Calculate the reaction of the hoop on the particle by means of the Lagrangian multipliers and Lagrange's equations. Find the height at which the particle falls off.

4.7. Consider a car that is driven up an inclined slope (Fig. P4.7). With the use of constrained Lagrangian equations, find the equations of motion, and also find the power required to drive the car at the minimum speed. Make assumptions necessary to simplify the problem.

4.8. A circular loop of wire is located in the x–y plane, with one point on it fixed at the origin and its center on the y axis; the radius varies in time according to $r = a + bt^2$, where a and b are constants. Find the equations of motion for a bead of mass m sliding smoothly on the wire and the normal force of wire on the bead (expressed as a function of an appropriate angular coordinate and its time derivative).

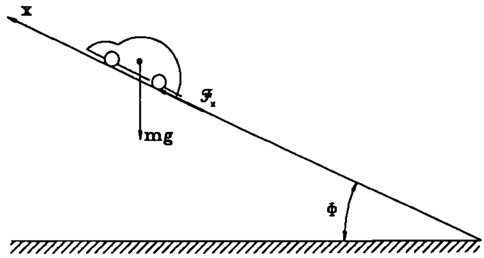

Fig. P4.7

4.9. Find the geodesic of a sphere.

4.10. The ends of a uniform inextensible string of length ℓ are connected to two points fixed at the same level, a distance $2a$ apart. Find the curve along which the string must hang if it is to have its center of mass as low as possible.

5
Rockets and Space Vehicles

IN this chapter we shall study the dynamics of rockets and space vehicles in detail. We begin the study with a single-stage rocket in Section 5.1. In this section, we discuss thrust, air drag, stability, equation of motion, and conditions at the time of burnout. Multistage rockets are studied in Section 5.2. Advantages of multistage design are explained. The method of Lagrangian multiplier is employed to achieve optimum design for a multistage rocket. A numerical example is given to demonstrate the advantages of multistage design.

The orbit of a space vehicle is studied in Section 5.3. The space vehicle is modeled as a particle in a central force field. Different orbits may be achieved with different amounts of total mechanical energies. Special emphasis is placed on elliptical orbits. Numerical examples are given to illustrate the relationship between the velocity and position of a space vehicle for getting into an elliptical orbit.

Continuous propulsion in a rocket is discussed in Section 5.4. Usually this type of propulsion is provided by an electrical system. Because the thrust from electrical propulsion is small compared to the weight of the rocket, small perturbation method is applied for solving the equations of motion. The advantage of analytical method is that parameters involved in the result are seen clearly.

Interplanetary orbits of a space vehicle are discussed in Section 5.5. The launching time is small compared to the period required for an interplanetary trip; therefore, the thrust and time for launching are considered as an impulse. The space vehicle in orbit is still modeled as a particle in central force field. Numerical results of different trajectories are collected in Table 5.2. A detailed calculation for an elliptical trajectory of a space probe traveling from Earth to Mars is given for this subject. Special attention is paid to the space probe when it reaches Mars. With a proper impulse to reduce the speed of the probe, it will get into a spiral orbit around Mars so that a long-time observation can be carried out.

5.1 Single-Stage Rockets

Rockets differ from air-breathing jet engines that burn fuel with surrounding air. Rockets are self-contained, carrying both fuel and oxidizer. To understand better, we must look into details about the forces acting on the rocket. In general, there are three forces: thrust, gravity, and air drag. In addition, during early development of the space program, many rocket launches failed at the launching pad. What were the reasons behind this? Finally, we want to know what are the conditions of the rocket when fuel and oxidizer are burned. All of these interesting subjects will be explored in this section.

Thrust

The thrust of a rocket can be determined by examining the performance of a rocket under static tests. The rocket is arranged schematically as shown in Fig. 5.1.

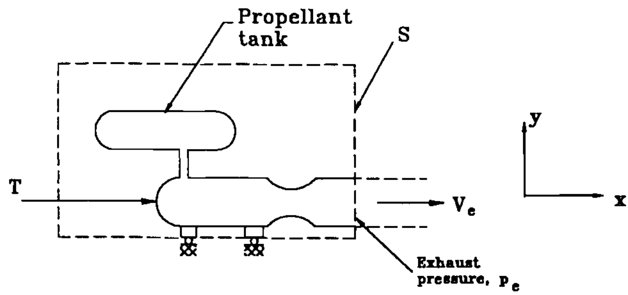

Fig. 5.1 Rocket under static tests.

Consider a stationary control surface that intersects the jet through the exit plane of the nozzle. Positive thrust acts in the direction opposite to V_e. The momentum equation for such a control volume is

$$\sum F = \frac{\mathrm{d}}{\mathrm{d}t}\int_v \rho V \mathrm{d}v + \int_A \rho V(V_r \cdot \mathrm{d}A) \tag{5.1}$$

$$\sum F = (T + A_e P_a - A_e P_e)i \tag{5.2}$$

where V is the velocity of fluid, V_r is the relative velocity between the fluid and the control volume, P_a is the ambient pressure, P_e is the exhaust pressure, and A_e is the exit area of the nozzle. The first term on the right-hand side of Eq. (5.1) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v \rho V \mathrm{d}v = 0$$

because $V = 0$. The second term is

$$\int_A \rho V(V_r \cdot \mathrm{d}A) = \dot{m} V_e i$$

Therefore, we have the thrust,

$$T = \dot{m} V_e + A_e(P_e - P_a) \tag{5.3}$$

Gravity

Because the gravitational force is inversely proportional to the distance squared between the center of the Earth and the mass center of the rocket, the gravity at different heights above the surface of the Earth can be expressed simply as

$$g = g_0 \left(\frac{R_0}{R_0 + h}\right)^2 \tag{5.4}$$

where g_0 is the gravity at the surface of the Earth, R_0 is the average radius of the Earth, 6,371.23 km, and h is the distance from the surface of the Earth.

Air Drag

The air drag acting on the rocket can be estimated by

$$D = C_d \frac{1}{2} \rho v^2 A_f \tag{5.5}$$

where C_d is the drag coefficient in the order of 0.1, ρ is the air density, (0.075 lbm/ft^3 at sea level), v is the rocket velocity, and A_f is the frontal cross-sectional area of the rocket.

From Eq. (5.5), it is seen easily that the drag is a function of velocity and density of air. At the begining of the rocket journey, the velocity is very small; later on the density becomes very small. The atmospheric density is reduced to 1% of its sea-level value at an altitude of 100,000 ft. Therefore, the drag value is always much less than the thrust of a rocket. Because of that, in the estimate of conditions after burning of fuel and oxidizer, the drag term is often omitted.

Stability

At the beginning of the launching process or shortly after the rocket leaves the launching pad, the forces acting on the rocket actually are thrust and gravity. It is easily seen that the thrust is produced by the exhaust gas at the exit of the nozzle. The sum of all the momentums of leaving particles $\sum_i \dot{m}_i V_{ei}$ is the major contribution to the thrust. The other part of the thrust is from pressure, which contributes a small fraction of the thrust. The vector sum of all $\dot{m}_i V_{ei}$ will locate the center of application of the thrust, C.T. as shown in Fig. 5.2. If C.T. is above the center of mass of the rocket, the situation is stable. Otherwise, the forces are not stable. The rocket most likely fails to be launched.

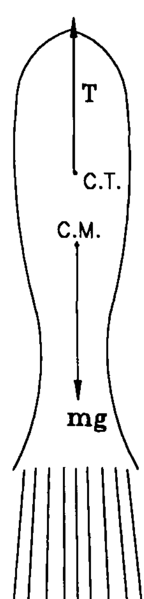

Fig. 5.2 Stability.

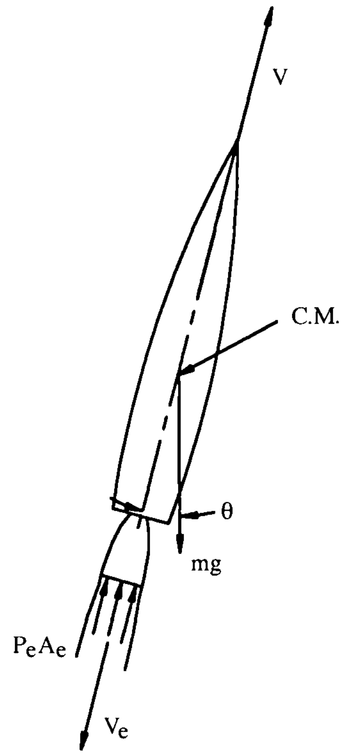

Fig. 5.3 Motion of a rocket in gravitational field.

One remark ought to be added here: the exit velocity V_e is obviously very important to the location of C.T. However, the exit velocity is not determined completely by the contour of the nozzle. The expansion wave of the flow usually occuring at the corner of the exit will change the direction and magnitude of the exit velocity. Details of these topics are beyond the scope of this book.

Conditions of the Rocket at the Time of Burnout

Consider that a rocket is launched at an angle of θ with the gravitational force as shown in Fig. 5.3. The equation of motion for the rocket along the axis of the rocket can be written as

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = T - D - mg\cos\theta \tag{5.6}$$

Note that as $T - D =$ net thrust denoted by F, the preceding equation agrees well with Eq. (2.19) $\times \sin\theta +$ Eq. (2.20) $\times \cos\theta$. Considering T, D, and g in precise form, Eq. (5.6) becomes

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = (P_e - P_a)A_e + \dot{m}V_e - c_D\frac{1}{2}\rho v^2 A_f - mg_0\frac{R_0^2}{(R_0 + h)^2}\cos\theta \tag{5.7}$$

This equation can be integrated numerically, as shown previously in the integration of Eqs. (2.19) and (2.20). However, if only the major terms are kept in the equation, we can have the equation simplified to

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = \dot{m}V_e - mg_0\cos\theta \tag{5.8}$$

Integrating the equation, we find

$$V_b = V_e \ln\frac{m_0}{m_b} - g_0(\cos\theta)_{\mathrm{av}}t_b \tag{5.9}$$

where $()_b$ is the quantity at the time of burnout, m_0 is the initial mass of the rocket, and $(\cos\theta)_{\mathrm{av}}$ is the integrated average value of $\cos\theta$. For a vertical flight the velocity is

$$V = V_e \ln\frac{m_0}{m} - g_0 t \tag{5.10}$$

where $m = m_0 - \dot{m}t$.

The altitude attained by the rocket at burnout is

$$h_b = \int_0^{t_b} v\,\mathrm{d}t = -V_e t_b \frac{\ln(m_0/m_b)}{(m_0/m_b) - 1} + V_e t_b - \frac{1}{2}g_0 t_b^2 \tag{5.11}$$

To see clearly the advantage of multistage design for rocket and save some writing, let us introduce mass ratio R as

$$R = \frac{m_0}{m_b} \tag{5.12}$$

payload ratio

$$\lambda = \frac{\text{payload mass}}{\text{mass of propellant and structure}} = \frac{m_L}{m_p + m_s} \tag{5.13}$$

and the structure coefficient ϵ as

$$\epsilon = \frac{\text{structure mass}}{\text{mass of propellant and structure}} = \frac{m_s}{m_p + m_s} = \frac{m_b - m_L}{m_0 - m_L} \tag{5.14}$$

From the preceding equations it is clearly implied that

$$m_0 = m_L + m_p + m_s \tag{5.15}$$

and

$$m_b = m_L + m_s \tag{5.16}$$

Combining the expressions already introduced, the mass ratio can be written as

$$R = \frac{1+\lambda}{\epsilon+\lambda} \tag{5.17}$$

and the terminal velocity of the rocket at the burnout is

$$V_f = V_e \ln R - g_0 t_b = V_e \ln \frac{1+\lambda}{\epsilon+\lambda} - g_0 t_b \tag{5.18}$$

5.2 Multistage Rockets

From past observations, many rockets are designed in two or three stages. Theoretically speaking more stages always will make the terminal velocity higher. However, the practical design problem also must be considered carefully. The optimization of multistage rockets with respect to the distribution of mass has been treated in a number of interesting papers.* To simplify the problem, let us only consider the first term on the right-hand side of Eq. (5.18) and write ΔV_i for the increment of velocity of the ith stage of the rocket so that

$$\Delta V_i = V_e \ln \frac{1+\lambda_i}{\epsilon_i+\lambda_i} \tag{5.19}$$

The final velocity of nth stage is then

$$V_n = \sum_i \Delta V_i = V_e \sum_{i=1}^{n} \ln \frac{1+\lambda_i}{\epsilon_i+\lambda_i}$$

or

$$\frac{V_n}{V_e} = \sum_{I}^{n} \ln \frac{1+\lambda_i}{\epsilon_i+\lambda_i} = \sum_{i=1}^{n} F(\lambda_i) \tag{5.20}$$

Here we can maximize V_n/V_e by adjusting the value of λ_i. On the other hand, for each stage, we have

$$\lambda_i = \frac{m_{0(i+1)}}{m_{0i} - m_{0(i+1)}}$$

$$\frac{m_{0i}}{m_{0(i+1)}} = \frac{1+\lambda_i}{\lambda_i}$$

where m_{0i} is the initial mass of the ith stage of the rocket. That means

$$\frac{m_{01}}{m_L} = \frac{m_{01}}{m_{02}} \cdot \frac{m_{02}}{m_{03}} \cdots \frac{m_{0n}}{m_L} = \prod_{i=1}^{n} \left(\frac{1+\lambda_i}{\lambda_i} \right)$$

*Hill, P. G., and Peterson, C. R., *Mechanics and Thermodynamics of Propulsion,* McGraw-Hill, New York, 1983.

or

$$\frac{m_L}{m_{01}} = \prod_{i=1}^{n} \left(\frac{\lambda_i}{1+\lambda_i} \right) \tag{5.21}$$

Taking logarithmic form of the preceding equation, we obtain

$$\ell n \frac{m_L}{m_{01}} = \sum_{i=1}^{n} \ell n \left(\frac{\lambda_i}{1+\lambda_i} \right) = \sum_{i=1}^{n} G(\lambda_i) \tag{5.22}$$

which actually serves as a constraint equation for adjusting λ_i, because for a given design, the payload and the initial mass must be specified. Therefore, we reach the point that (V_n/V_e) is to be maximized but subjected to the constraint equation of (5.22). This is a typical problem for the use of the Lagrange multiplier. Consider

$$L(\lambda_i) = F(\lambda_i) + \alpha G(\lambda_i) \tag{5.23}$$

where α is the Lagrange multiplier. Taking the derivative of Eq. (5.23) with respect to λ_i and setting it to zero, we find

$$\frac{\partial L}{\partial \lambda_i} = \frac{\partial F}{\partial \lambda_i} + \alpha \frac{\partial G}{\partial \lambda_i} = 0$$

$$\frac{1}{1+\lambda_i} - \frac{1}{\epsilon + \lambda_i} + \frac{\alpha}{\lambda_i} - \frac{\alpha}{1+\lambda_i} = 0$$

which can be simplified to

$$\lambda_i = \frac{\alpha \epsilon_i}{1 - \alpha - \epsilon_i} \tag{5.24}$$

Then from Eq. (5.21), the Lagrange multiplier α can be determined by

$$\frac{m_L}{m_{01}} = \prod_{i=1}^{n} \left(\frac{\epsilon_i}{1-\epsilon_i} \right) \left(\frac{\alpha}{1-\alpha} \right) \tag{5.25}$$

or

$$\alpha = 1 \Bigg/ \left\{ 1 + \left[\frac{m_{01}}{m_L} \prod_{i=1}^{n} \left(\frac{\epsilon_i}{1-\epsilon_i} \right) \right]^{\frac{1}{n}} \right\} \tag{5.26}$$

Then the value of λ_i is determined by the value of α in Eq. (5.24).

Example 5.1

To illustrate the advantage of multistage design, let us compare the terminal velocity of a single-stage rocket to that of a three-stage rocket. Suppose that the payload is 500 kg, the initial mass is 7500 kg, and the exhaust velocity is 3000 mps. The structure mass is 1000 kg.

Solution. For the single-stage rocket

$$\epsilon = \frac{m_s}{m_{01} - m_L} = \frac{1000}{7500 - 500} = 0.143$$

$$\lambda = \frac{m_L}{m_{01} - m_L} = \frac{500}{7500 - 500} = 0.0714$$

$$v_f = V_e \ln \frac{1+\lambda}{\epsilon+\lambda} = 3000 \ln \frac{1 + 0.0714}{0.143 + 0.0714} = 4827 \text{ m/s}$$

For the three-stage rocket, by assuming

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.143$$

and from Eq. (5.26), we obtain

$$\alpha = \frac{1}{1 + (7500/500)^{\frac{1}{3}}(0.143/0.857)} = 0.70846$$

Using Eq. (5.24), we find

$$\lambda = \frac{\alpha\epsilon}{1 - \alpha - \epsilon} = \frac{0.70846 \times 0.143}{1 - 0.70846 - 0.143} = 0.68203$$

The terminal velocity at burnout is then obtained from Eq. (5.20):

$$\begin{aligned} V_f &= 3V_e \ln\left(\frac{1+\lambda}{\epsilon+\lambda}\right) \\ &= 9000 \ln\left(\frac{1.68203}{0.143 + 0.68203}\right) \\ &= 6411 \text{ m/s} \end{aligned}$$

Certainly, this velocity is much higher than the velocity of the single-stage rocket.

5.3 Motion of a Particle in Central Force Field

Consider a system of two particles with mass m_1 and m_2. Let the center of mass m_2 be at the origin of x–y plane. This plane contains the trajectory of m_1. Furthermore, let us consider the case $m_2 \gg m_1$ and write M for m_2, m for m_1. With the use of polar coordinates (r, θ), Lagrange's function for m is

$$L = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

Then the equations of motion for m are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{5.27}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 0 \tag{5.28}$$

From Eq. (5.28), we obtain the momentum in θ direction as

$$mr^2\dot{\theta} = \mathcal{L} \tag{5.29}$$

where $\mathcal{L}$ is a constant. This means that, as the particle moves in a central force field, its angular momentum is constant. With the information of Eq. (5.29), Eq. (5.27) becomes

$$m\ddot{r} - \frac{\mathcal{L}^2}{mr^3} = -\frac{\partial V}{\partial r} = F(r) \tag{5.30}$$

$F(r)$ is the force in the r direction. Because the potential energy of the particle is a function of r only, the force is a function of r. Equation (5.30) actually defines $r(t)$.

To solve Eq. (5.30), we use the inverse square law for the force, i.e.,

$$F(r) = -\frac{GMm}{r^2} \tag{5.31}$$

where G is the universal gravitational constant $= 6.670 \times 10^{-11}$ $\mathrm{N \cdot m^2/kg^2}$. Because M and m are known quantities, the force may be written simply as

$$F(r) = -\frac{k}{r^2}$$

where $k = GMm$. Now the equation becomes

$$m\ddot{r} - \frac{\mathcal{L}^2}{mr^3} = -\frac{k}{r^2} \tag{5.32}$$

To solve this equation analytically, we rearrange the equation. Because

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta} = \frac{\mathcal{L}}{mr^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{\mathcal{L}}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \left(\frac{\mathcal{L}}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)\left(\frac{\mathcal{L}}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)$$

Eq. (5.32) now becomes

$$\frac{\mathcal{L}}{r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\left(\frac{\mathcal{L}}{mr^2}\right)\frac{\mathrm{d}r}{\mathrm{d}\theta}\right]-\frac{\mathcal{L}^2}{mr^3}=-\frac{k}{r^2} \tag{5.33}$$

The preceding equation can be simplified further by changing the variable. Let $\mu = 1/r$, then

$$\frac{\mathrm{d}\mu}{\mathrm{d}\theta}=-\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}$$

And we can write Eq. (5.33) as

$$-\mathcal{L}\mu^2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathcal{L}}{m}\frac{\mathrm{d}\mu}{\mathrm{d}\theta}\right)-\frac{\mathcal{L}^2}{m}\mu^3=-k\mu^2$$

Simplifying leads to

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\theta^2}+\mu=\frac{mk}{\mathcal{L}^2} \tag{5.34}$$

Without losing generalization, we can write the solution of Eq. (5.34) as

$$\mu=\frac{mk}{\mathcal{L}^2}[1+\varepsilon\cos(\theta-\theta')] \tag{5.35}$$

where ε and θ' are arbitrary constants of integration. To determine these constants, we put back the symbol r for $1/\mu$.

$$r=\frac{\mathcal{L}^2/(mk)}{1+\varepsilon\cos(\theta-\theta')} \tag{5.36}$$

Differentiating Eq. (5.36) with respect to θ, we find

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{\varepsilon\mathcal{L}^2}{mk}\frac{\sin(\theta-\theta')}{[1+\varepsilon\cos(\theta-\theta')]^2} \tag{5.37}$$

On the trajectory of m, there is a point called an apsidal point. At such a point, the r is not changed as θ changes. Let us choose the (r,θ) coordinates in such a way that $\theta-\theta'=0$ at one apsidal point. On the other hand, using Eqs. (5.36) and (5.37), we have

$$\dot r=\frac{\mathcal{L}}{mr^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{\mathcal{L}}{mr^2}\left(\frac{\varepsilon\mathcal{L}^2}{mk}\right)\frac{\sin(\theta-\theta')}{[1+\varepsilon\cos(\theta-\theta')]^2}$$

$$=\frac{\mathcal{L}}{m}\left(\frac{mk}{\mathcal{L}^2}\right)^2\left(\frac{\varepsilon\mathcal{L}^2}{mk}\right)\sin(\theta-\theta')$$

$$\dot r=\frac{\varepsilon k}{\mathcal{L}}\sin(\theta-\theta') \tag{5.38}$$

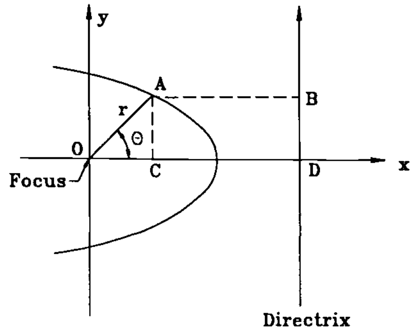

Fig. 5.4 Geometry of a conic curve.

The total energy of m can be written as

$$E = T + V = \frac{1}{2}m\dot{r}^2 + \frac{\mathcal{L}^2}{2mr^2} - \frac{k}{r} = (\varepsilon^2 - 1)\frac{mk^2}{2\mathcal{L}^2}$$

Equations (5.36) and (5.38) are used in the process deriving the preceding equation. Hence

$$\varepsilon = \sqrt{1 + \frac{2\mathcal{L}^2 E}{mk^2}} \tag{5.39}$$

Now the trajectory equation is

$$r = \frac{(\mathcal{L}^2/mk)}{1 + \varepsilon \cos\theta} \tag{5.40}$$

To understand the meaning of Eq. (5.40), let us review a part of analytical geometry for conic curves. A conic curve is defined as the locus of a point moving such that the ratio of its distance from a fixed point, the focus, to its distance from a fixed line, the directrix, is a constant ε. From Fig. 5.4, we have

$$\varepsilon = \frac{r}{AB}$$

or

$$r = \varepsilon(AB) = \varepsilon(CD) = \varepsilon(OD - r\cos\theta)$$

Rearranging leads to

$$r(1 + \varepsilon\cos\theta) = \varepsilon \cdot OD = \text{const} = C$$

Therefore

$$r = \frac{C}{1 + \varepsilon\cos\theta} \tag{5.41}$$

Table 5.1 Different values of ϵ and E for different orbits

Eccentricity, ε	Energy, E	Type of orbit
>1	>0	Hyperbola
$=1$	$=0$	Parabola
<1 but >0	<0	Ellipse
$=0$	$-mk^2/(2\mathcal{L}^2)$	Circle

Compare this equation with Eq. (5.40); we find

$$C = \mathcal{L}^2/(mk)$$

That means the orbit of the particle in a central force field can be one of the conic curves. The focus is the center of central force field. Different conic curves result from different values of ε, which is called eccentricity. Because E is directly related to ε, different orbits for different ε and E are given in Table 5.1.

Because the total energy of the particle m dictates the type of orbit, let us look into the meaning of $E < 0$, i.e.,

$$E = T + V < 0$$

$$T < -V(r) = \frac{k}{r} = \frac{GMm}{r}$$

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) < \frac{GMm}{r}$$

or

$$\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) < \frac{GM}{r}$$

The preceding equation says for an elliptical orbit, the velocity of the particle must be less than $\sqrt{2GM/r}$. As the velocity reaches the limiting value of $\sqrt{2GM/r}$, the particle will get on the parabolic orbit and will not come back. Hence this velocity is termed the escape velocity:

$$V_{\text{esc}} = \sqrt{\frac{2GM}{r}} \tag{5.42}$$

On the other hand, for a circular orbit, $\varepsilon = 0$. Let us examine the meaning of $\varepsilon = 0$. As

$$\varepsilon = \sqrt{1 + \frac{2\mathcal{L}^2 E}{mk^2}} = 0$$

that means

$$1 + \frac{2\mathcal{L}^2 E}{mk^2} = 0$$

$$E = -\frac{mk^2}{2\mathcal{L}^2}$$

$$T + V = -\frac{mk^2}{2\mathcal{L}^2}$$

$$\frac{1}{2}m(\dot{r} + r^2\dot{\theta}^2) - \frac{GMm}{r} = -\frac{1}{2}m\frac{(GM)^2}{(r^2\dot{\theta})^2}$$

For a circular orbit, $\dot{r} = 0$, and simplifying the preceding equation, we find

$$\frac{1}{2}r^2\dot{\theta}^2 = \frac{GM}{r} - \frac{(GM)^2}{2(r^2\dot{\theta})^2}$$

Using $v = r\dot{\theta}$, we have

$$\frac{1}{2}v^2 = \frac{GM}{r} - \frac{(GM)^2}{2r^2v^2}$$

$$(v^2 - GM/r)^2 = 0$$

$$v_{\text{cir}} = \sqrt{GM/r} \tag{5.43}$$

That means

$$v_{\text{esc}} = \sqrt{2}v_{\text{cir}}$$

and for an elliptical orbit, the velocity must satisfy the condition

$$v_{\text{cir}} < v_{\text{ell}} < v_{\text{esc}}$$

or

$$\sqrt{GM/r} < v_{\text{ell}} < \sqrt{2GM/r} \tag{5.44}$$

Just to have some feeling of the velocity of a planet on a circular orbit, let us calculate the velocity of the Earth around the sun. We have

$$v_{\text{Earth}} = \sqrt{\frac{GM_{\text{sun}}}{R_{\text{Earth}}}}$$

$$M_{\text{sun}} = 1.9866158 \times 10^{30}\ \text{kg}$$

$$R_{\text{Earth}} = 1.495 \times 10^{11}\ \text{m}$$

$$v_{\text{Earth}} = \sqrt{\frac{6.67 \times 10^{-11} \times 1.9866158 \times 10^{30}}{1.495 \times 10^{11}}} = 29771.4\ \text{m/s}$$

$$\simeq 30\ \text{km/s}$$

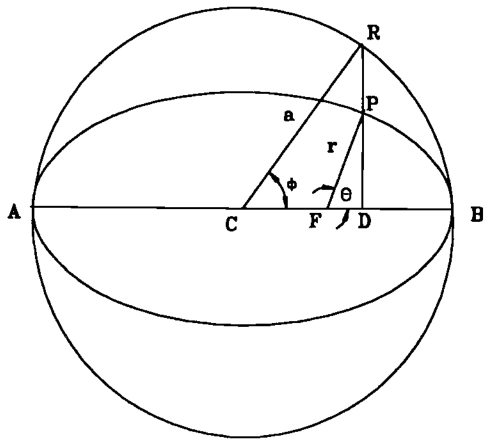

Fig. 5.5 Ellipse and auxiliary circle.

Elliptical Orbits

In studying the orbits of satellites around the Earth, we are interested more in elliptical orbits. Let us look further into details about them. From Fig. 5.5, we have

$$FC = CD - FD = a\cos\phi - r\cos\theta$$

Also, we have

$$FC = a - BF = a - r_{\min} = a - \frac{\mathcal{L}^2}{mk(1+\varepsilon)}$$

Because

$$r_{\max} = \frac{\mathcal{L}^2}{mk(1-\varepsilon)}$$

$$r_{\min} + r_{\max} = \frac{\mathcal{L}^2}{mk}\left[\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon}\right]$$

$$= \frac{2\mathcal{L}^2}{mk(1-\varepsilon^2)} = 2a$$

we find

$$a = \frac{\mathcal{L}^2}{mk(1-\varepsilon^2)} \tag{5.45}$$

$$r_{\min} = a(1-\varepsilon) \tag{5.46}$$

and

$$FC = a - a(1 - \varepsilon) = a\varepsilon$$

$$= a\cos\phi - r\cos\theta = a\cos\phi - \frac{\mathcal{L}^2/mk - r}{\varepsilon}$$

$$= a\cos\phi - \frac{a(1-\varepsilon^2)}{\varepsilon} + \frac{r}{\varepsilon}$$

$$r = a(1 - \varepsilon\cos\phi) \tag{5.47}$$

So far, we have found the orbital equation $r = r(\theta)$ or $r = r(\phi)$, but there is no equation to have time t explicitly involved. To relate ϕ to the time, let us start from the total energy E:

$$E = \frac{m}{2}\dot{r}^2 + \frac{\mathcal{L}^2}{2mr^2} - \frac{k}{r}$$

so that

$$\dot{r} = \sqrt{\frac{2}{m}\left(E + \frac{k}{r} - \frac{\mathcal{L}^2}{2mr^2}\right)}, \qquad \mathrm{d}r \bigg/ \sqrt{\frac{2}{m}\left(E + \frac{k}{r} - \frac{\mathcal{L}^2}{2mr^2}\right)} = \mathrm{d}t \tag{5.48}$$

Because $E = \text{const}$, it remains the same at any value of r. Let us consider E at $r = r_{\min}$.

$$E = T + V = \frac{1}{2}mr_{\min}^2\dot{\theta}^2 - \frac{k}{r_{\min}}$$

$$= \frac{1}{2}\frac{\mathcal{L}^2}{mr_{\min}^2} - \frac{k}{r_{\min}}$$

$$= \frac{1}{2m}\left\{\mathcal{L}^2 \bigg/ \left[\frac{\mathcal{L}^2}{mk(1+\varepsilon)}\right]^2\right\} - \left[k \bigg/ \frac{\mathcal{L}^2}{mk(1+\varepsilon)}\right]$$

$$= \frac{mk^2}{2\mathcal{L}^2}(\varepsilon^2 - 1) = -\frac{k}{2a} \tag{5.49}$$

Substituting Eq. (5.49) into Eq. (5.48), we have

$$\mathrm{d}t = r\,\mathrm{d}r \bigg/ \sqrt{\frac{2}{m}\left(-\frac{k}{2a}r^2 + kr - \frac{\mathcal{L}^2}{2m}\right)} = \frac{\sqrt{ma/k}\,r\,\mathrm{d}r}{\sqrt{(a\varepsilon)^2 - (r-a)^2}}$$

$$= \frac{\sqrt{ma/k}\,a(1 - \varepsilon\cos\phi)a\varepsilon\sin\phi\,\mathrm{d}\phi}{\sqrt{(a\varepsilon)^2 - (-a\varepsilon\cos\phi)^2}} = \frac{\sqrt{ma}}{k}a(1 - \varepsilon\cos\phi)\,\mathrm{d}\phi \tag{5.50}$$

Furthermore, because $mr^2\dot{\theta} = \mathcal{L} = \text{const}$ or $r^2\dot{\theta} = \text{const}$,

$$\tfrac{1}{2}r^2 \mathrm{d}\theta = \mathrm{d}A$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \text{const} = \frac{\mathcal{L}}{2m} = \frac{\pi ab}{T}$$

This is known as Kepler's law of areas. Where T is the period of the motion, πab is the area of the ellipse. Making use of the relations from analytical geometry,

$$b = a\sqrt{1-\varepsilon^2}$$

$$\varepsilon = (c/a)$$

we find

$$T = \frac{2\pi abm}{\mathcal{L}} = 2\pi a^2\sqrt{1-\varepsilon^2}\frac{m}{\mathcal{L}}$$

$$= 2\pi a^2 \frac{\sqrt{1-\varepsilon^2}m}{\sqrt{mka(1-\varepsilon^2)}} = 2\pi\sqrt{\frac{ma^3}{k}} \tag{5.51}$$

Using Eq. (5.51) in Eq. (5.50), we obtain

$$\mathrm{d}t = \frac{T}{2\pi}(1-\varepsilon\cos\phi)\mathrm{d}\phi$$

Therefore

$$\frac{2\pi t}{T} = \phi - \varepsilon\sin\phi \tag{5.52}$$

Collecting all the results together, now we have

$$r = \frac{\mathcal{L}^2/(mk)}{1+\varepsilon\cos\theta}, \qquad \varepsilon = \sqrt{1+\frac{2\mathcal{L}^2E}{mk^2}}$$

$$r = a(1-\varepsilon\cos\phi), \qquad a = \frac{\mathcal{L}^2}{mk}\frac{1}{(1-\varepsilon^2)}$$

$$\frac{2\pi t}{T} = \phi - \varepsilon\sin\phi, \qquad T = 2\pi\sqrt{\frac{ma^3}{k}}$$

Example 5.2

Consider a particle that is moving in an elliptical orbit about a fixed focus because of an inverse-square law of attraction. 1) Find the points in the orbit at which the magnitude of the radial velocity $\dot{r}$ is maximum, and 2) prove that the possible values of corresponding $\dot{\theta}$ are

$$\sqrt{\frac{k}{mr^3}\left(1-\frac{b}{a}\right)} < \dot{\theta} < \sqrt{\frac{k}{mr^3}\left(1+\frac{b}{a}\right)}$$

Solution. 1) Rewrite the equations of motion for a particle in an elliptical orbit:

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{k}{r^2} \tag{5.53}$$

$$mr^2\dot{\theta} = \mathcal{L} \tag{5.54}$$

As $\dot{r} \to \dot{r}_{\max}, \ddot{r} = 0$, then from Eq. (5.53), we have

$$mr\dot{\theta}^2 = \frac{k}{r^2}$$

$$mr\left(\frac{\mathcal{L}}{mr^2}\right)^2 = \frac{k}{r^2}$$

or

$$r = \frac{\mathcal{L}^2}{mk} \tag{5.55}$$

On the other hand,

$$r = \frac{\mathcal{L}^2/(mk)}{1+\varepsilon\cos\theta}$$

$$\dot{r} = \frac{k\varepsilon\sin\theta}{\mathcal{L}}$$

$$\ddot{r} = \frac{k\varepsilon}{\mathcal{L}}\cos\theta\dot{\theta} = 0$$

That means $\theta = \pi/2$ or $3\pi/2$:

$$\dot{r}_{\max} = \frac{k\varepsilon}{\mathcal{L}} \quad \text{at} \quad \theta = \frac{\pi}{2}, \qquad r = \frac{\mathcal{L}^2}{mk} \tag{5.56}$$

2) For a particle in an elliptical orbit,

$$E = \frac{1}{2}m(\dot{r} + r^2\dot{\theta}^2) - \frac{k}{r} < 0$$

$$\dot{\theta}^2 < \frac{1}{r^2}\left(\frac{2k}{mr} - \dot{r}^2\right) = \frac{2k}{mr^3} - \left(\frac{\varepsilon k}{\mathcal{L}}\right)^2\frac{1}{r^2} = \frac{2k}{mr^3} - \frac{\varepsilon^2k^2}{m^2r^6\dot{\theta}^2}$$

or

$$\dot{\theta}^4 - \frac{2k}{mr^3}\dot{\theta}^2 + \frac{\varepsilon^2 k^2}{m^2 r^6} < 0 \tag{5.57}$$

Consider

$$f(\dot{\theta}^2) = \dot{\theta}^4 - \frac{2k}{mr^3}\dot{\theta}^2 + \frac{\varepsilon^2 k^2}{m^2 r^6} = 0$$

$$\dot{\theta}^2 = \frac{1}{2}\left[\frac{2k}{mr^3} \pm \sqrt{\frac{4k^2}{m^2 r^6} - \frac{4\varepsilon^2 k^2}{m^2 r^6}}\right]$$

$$= \frac{k}{mr^3}[1 \pm \sqrt{1-\varepsilon^2}] = \frac{k}{mr^3}\left(1 \pm \frac{b}{a}\right) = \begin{cases}\dot{\theta}_1^2 \\ \dot{\theta}_2^2\end{cases} \tag{5.58}$$

Therefore, choosing the values of $\dot{\theta}$ between the two roots from Eq. (5.58), Eq. (5.57) is satisfied, i.e.,

$$f(\dot{\theta}^2) = \left(\dot{\theta}^2 - \dot{\theta}_1^2\right)\left(\dot{\theta}^2 - \dot{\theta}_2^2\right) < 0$$

Example 5.3

A weather satellite is to be launched. The requirement of such a satellite is that it must stay above the same point on the surface of the Earth all the time. Determine the radius of the circular orbit above a point located along the line from the center of Earth perpendicular to the Earth's rotating axis with mass of Earth $= 5.975 \times 10^{24}$ kg.

Solution. For a circular orbit, the velocity of the satellite is

$$v = \sqrt{GM/r}$$

Because the satellite must be moving with $v = r\omega$, where ω is the rotating speed of the Earth,

$$\omega = \frac{2\pi}{24 \times 60 \times 60} = 7.2722 \times 10^{-5} \text{ rad/s}$$

$$\omega r = \sqrt{GM/r}$$

we find

$$r = \frac{(GM)^{1/3}}{\omega^{2/3}} = \frac{(6.67 \times 10^{-11} \times 5.975 \times 10^{24})^{1/3}}{(7.2722 \times 10^{-5})^{2/3}}$$

$$= 4.22387 \times 10^4 \qquad \text{(km)}$$

Example 5.4

A satellite enters its orbit at a velocity of 8045 m/s at an altitude of 644 km. The velocity is parallel to the Earth's surface. Find the equation for the orbit and the maximum altitude from the Earth's surface the satellite will reach. The average radius of Earth is 6436 km and the mass of Earth is 5.975×10^{24} kg.

Solution. From the given data, we have

$$r = 644{,}000 + 6{,}436{,}000 = 7{,}080{,}000 \qquad \text{(m)}$$

$$\mathcal{L} = mr^2\dot{\theta} = mrv = m(7{,}080{,}000)(8045)$$

$$= m(5.69586 \times 10^{10}) \qquad (\text{kg} \cdot \text{m}^2/\text{s})$$

$$E = \frac{\mathcal{L}^2}{2mr^2} - \frac{k}{r}$$

$$k = GMm = (6.67 \times 10^{-11})(5.975 \times 10^{24})m$$

$$= m(3.9853 \times 10^{14}) \qquad (\text{N} \cdot \text{m}^2)$$

$$E = \frac{(m5.69586 \times 10^{10})^2}{2m(7{,}080{,}000)^2} - \frac{m(3.9853 \times 10^{14})}{7{,}080{,}000}$$

$$= -m(23{,}928{,}536) \qquad (\text{N} \cdot \text{m})$$

$$\varepsilon = \sqrt{1 + \frac{2\mathcal{L}^2 E}{mk^2}} = \sqrt{1 - \frac{2(m5.69586 \times 10^{10})^2 m(23{,}928{,}536)}{m(m3.9853 \times 10^{14})^2}} = 0.1498$$

$$\mathcal{L}^2/(mk) = \frac{(m5.69586 \times 10^{10})^2}{m^2(3.9853 \times 10^{14})} = 8{,}140{,}622 \qquad \text{(m)}$$

Hence, the orbital equation is

$$r = \frac{\mathcal{L}^2/(mk)}{1 + \varepsilon\cos\theta} = \frac{8140622}{1 + 0.1498\cos\theta}$$

$$r_{\max} = 9{,}574{,}950 \qquad \text{(m)} \qquad \text{as } \theta = \pi$$

The maximum altitude is

$$h_{\max} = r_{\max} - r_{\text{Earth}} = 9{,}574{,}950 - 6{,}436{,}000$$

$$= 3{,}138{,}950 \qquad \text{(m)}$$

$$= 3139 \qquad \text{(km)}$$

5.4 Space Vehicle with Electrical Propulsion (equations solved by small perturbation method)

Electrical propulsion systems are known very low in thrust as compared to the gravity of the space vehicle at the Earth's surface. Because electrons are emitted from the sun constantly, space vehicles can collect electrons in orbit and the electrical propulsion system can function properly while the vehicle is traveling in space. Consider that a low thrust is oriented along the tangential direction of the orbit. The equations of motion are then

$$m(\ddot{r} - r\dot{\theta}^2) = -(k/r^2) \tag{5.59}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \Gamma_\theta \tag{5.60}$$

where Γ_θ is the electrical thrust in the tangential direction. To solve Eqs. (5.59) and (5.60), we introduce dimensionless variables as follows:

$$\rho = \frac{r}{r_0}, \qquad \tau = \sqrt{\frac{GM}{r_0^3}}t, \qquad \nu = \frac{\Gamma_\theta}{mg} = \frac{\Gamma_\theta r_0^2}{GMm} \tag{5.61}$$

where r_0 is the initial orbit radius and g is the gravitational acceleration at the initial orbit radius. Now we can write

$$\frac{dr}{dt} = \sqrt{\frac{GM}{r_0}}\frac{d\rho}{d\tau} \tag{5.62}$$

$$\frac{d\theta}{dt} = \sqrt{\frac{GM}{r_0^3}}\frac{d\theta}{d\tau} \tag{5.63}$$

$$\frac{d^2r}{dt^2} = \frac{GM}{r_0^2}\frac{d^2\rho}{d\tau^2} \tag{5.64}$$

With the use of these expressions, Eq. (5.60) becomes

$$\frac{d}{d\tau}\left(\rho^2\frac{d\theta}{d\tau}\right) = \nu\rho \tag{5.65}$$

and Eq. (5.59) becomes

$$\frac{d^2\rho}{d\tau^2} = \rho\left(\frac{d\theta}{d\tau}\right)^2 - \frac{1}{\rho^2} \tag{5.66}$$

Recall that the parameter ν is the ratio of the thrust from electrical rocket to the gravitational force at the beginning point while $r = r_0$ and is in the order of 10^{-3}. Therefore, this is a typical case to be solved by small perturbation method. To solve Eqs. (5.65) and (5.66), the initial conditions are assumed to be $\ddot{r} = \dot{r} = 0$ and $r = r_0$; the thrust is initiated at $t = 0$, and

$$\dot{\theta} = \sqrt{GM/r_0^3}$$

In dimensionless variables, that means

$$\rho = 1, \qquad \frac{d\rho}{d\tau} = 0, \qquad \frac{d^2\rho}{d\tau^2} = 0 \tag{5.67}$$

and

$$\frac{d\theta}{d\tau} = 1, \quad \theta = 0 \tag{5.68}$$

From Eq. (5.66) we get

$$\frac{d\theta}{d\tau} = \left[\frac{1}{\rho}\left(\frac{d^2\rho}{d\tau^2} + \frac{1}{\rho^2}\right)\right]^{\frac{1}{2}} \tag{5.69}$$

Substituting Eq. (5.69) into Eq. (5.65), we find

$$\frac{d}{d\tau}\left[\rho^3 \frac{d^2\rho}{d\tau^2} + \rho\right]^{\frac{1}{2}} = \nu\rho \tag{5.70}$$

To solve this equation, let us assume the solution can be expressed as

$$\rho = \rho_0 + \nu\rho_1 + \nu^2\rho_2 + \cdots \tag{5.71}$$

where ρ is a function of τ and has a magnitude in order of unity. With the use of Eq. (5.71), Eq. (5.70) becomes

$$\begin{aligned} &\frac{d}{d\tau}\left[(\rho_0 + \nu\rho_1 + \nu^2\rho_2 + \cdots)^3 \frac{d^2}{d\tau^2}(\rho_0 + \nu\rho_1 + \nu^2\rho_2 + \cdots)\right. \\ &\left. + (\rho_0 + \nu\rho_1 + \nu^2\rho_2 + \cdots)\right]^{\frac{1}{2}} \\ &= \nu\rho_0 + \nu^2\rho_1 + \cdots \end{aligned}$$

After carrying out the product in the preceding equation and breaking down the terms according to the orders of ν, we find the following equations. To the zeroth order of ν,

$$\frac{d}{d\tau}\left[\rho_0^3 \frac{d^2\rho_0}{d\tau^2} + \rho_0\right]^{\frac{1}{2}} = 0 \tag{5.72}$$

or

$$\rho_0^3 \frac{d^2\rho_0}{d\tau^2} + \rho_0 = c$$

The solution of this nonlinear differential equation is simply

$$\rho_0 = c$$

When the initial condition is applied, we find

$$\rho_0 = 1 \tag{5.73}$$

To the first order of ν, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}^2\rho_1}{\mathrm{d}\tau^2}+\rho_1\right)=1 \tag{5.74}$$

The solution of Eq. (5.74) is

$$\rho_1 = 2\tau - 2\sin\tau \tag{5.75}$$

Therefore, the solution of Eq. (5.70) up to the first order of ν is

$$\rho = 1 + \nu(2\tau - 2\sin\tau) + 0(\nu^2) \tag{5.76}$$

With the use of Eq. (5.76) in Eq. (5.69), the solution for $\theta(\tau)$ is obtained as

$$\theta = \tau - \nu(4\cos\tau + 1.5\tau^2 - 4) + 0(\nu^2) \tag{5.77}$$

From Eqs. (5.76) and (5.77) it is clear that the trajectory of the space vehicle is a spiral. The increment of the radius is proportional to the tangential thrust and the initial angular speed. The results are plotted in Fig. 5.6. Equations (5.65) and (5.66) can be solved numerically by the Runge–Kutta method. The disadvantage of numerical method is that the parameters involved cannot be seen immediately.

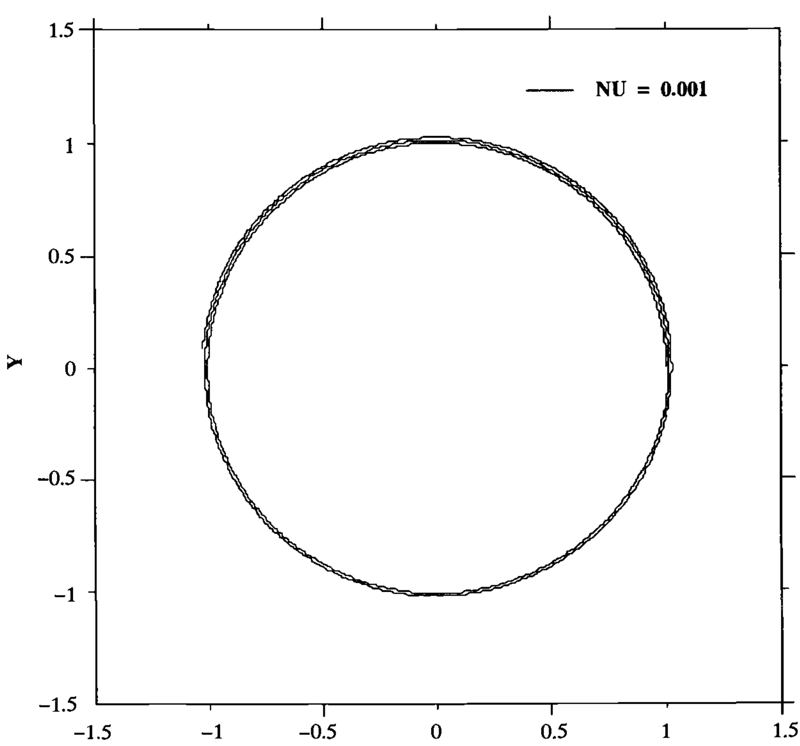

Fig. 5.6 Spiral orbit of an electrical rocket.

5.5 Interplanetary Trajectories

As a space vehicle moves in space, there is often more than one gravitational force acting on it. Therefore the equation of motion for the vehicle can be written as

$$m\ddot{\boldsymbol{r}} = \sum_i \boldsymbol{F}_i \tag{5.78}$$

where $\boldsymbol{F}_i = (GM_i m/|\boldsymbol{r} - \boldsymbol{r}_i|^3)(\boldsymbol{r}_i - \boldsymbol{r}) =$ the gravitational force from M_i. For example, when a spaceship travels from Earth to Mars, it is subjected to the gravitational forces from Earth, the sun, and Mars. However as the spaceship leaves Earth, the gravitational force will shift from Earth to the sun. To estimate the gravitational force from Earth, it is found that

$$F_{\text{Earth}} < \tfrac{1}{10} F_{\text{sun}}$$

as the spaceship moves away from the Earth by 1/1000 of the circumference of the Earth orbit around the sun. Hence it is not a bad approximation that the whole journey is divided into three segments; in each segment the ship is subjected to one gravitational force, so that the equations and solutions developed in Section 5.3 can be applied. The first segment is for the Earth's gravitational field, the second is for the sun's gravitational field, and the third for Mars's field. As the space vehicle reaches the escape velocity from the surface of Earth, it will stay in the Earth's circular orbit around the sun. In the second part of the journey, the trajectory is elliptical and is called transfer orbit. The last part of the journey is in Mars's circular orbit with the radius larger than that of Earth's circular orbit. Hohmann* studied the interplanetary trajectory first and used three impulses for the Earth to Mars journey. The total velocity increment for the vehicle to reach the Mars orbit is

$$\Delta U_{\text{Hohmann}} = U_{\text{esc}} + \Delta U_T + \Delta U_{\text{Mars}} \tag{5.79}$$

where U_{esc} is the velocity required for the vehicle to escape the gravitational field of Earth, ΔU_T is the increment of velocity as the vehicle moves from the Earth's circular orbit around the sun to the elliptical transfer orbit at $r = r_{\text{Earth}}$, and ΔU_{Mars} is the increment of velocity as the vehicle moves from the elliptical transfer orbit at $r = r_{\text{Mars}}$ to the circular orbit of Mars around the sun.

In the Hohmann treatment, the energy per unit mass required to put the vehicle into the transfer orbit is

$$E_{\text{Hofmann}} = E_{\text{esc}} + \tfrac{1}{2}(\Delta U_T)^2 = \tfrac{1}{2}(\Delta U_T)^2 \tag{5.80}$$

Because $E_{\text{esc}} =$ the escape energy $= \frac{1}{2}U_{\text{esc}}^2 - (GM_e/R_e) = 0$, where $R_e =$ radius of the Earth.

*Hohmann, W., "Die Erreichbarkeit der Himmelskorper (The Attainability of Heavenly Bodies)," NASA Technical Translations F-44, 1960.

Oberth* treated the problem slightly differently. He considered a higher terminal velocity for the space vehicle leaving the Earth, so that the vehicle can get into the transfer orbit directly from the first impulse of the rocket

$$E_{\text{Oberth}} = \frac{U^2_{\text{Oberth}}}{2} - \frac{GM_e}{R_e}$$

Therefore, the total energy per unit mass at the burning out time must equal to $(\Delta U_T)^2/2$, i.e.,

$$\frac{1}{2}U^2_{\text{Oberth}} - \frac{GM_e}{R_e} = \frac{1}{2}(\Delta U_T)^2$$

because

$$\frac{1}{2}U^2_{\text{esc}} = \frac{GM_e}{R_e}$$

Hence

$$U_{\text{Oberth}} = \sqrt{U^2_{\text{esc}} + (\Delta U_T)^2} \tag{5.81}$$

where $\Delta U_T = U_T - U_e$, U_T is the velocity of the spaceship on the transfer orbit, and U_e is its velocity on the circular orbit of Earth around the sun. From the total energy of the spaceship and Eq. (5.49), we have

$$E = \frac{mU_T^2}{2} - \frac{GM_{\text{sun}}m}{r_{\text{Earth}}} = -\frac{GM_{\text{sun}}m}{2a}$$

Hence,

$$U_T = \sqrt{GM_{\text{sun}}\left(\frac{2}{r_{\text{Earth}}} - \frac{1}{a}\right)}$$

With the velocity given in Eq. (5.81) as the first impulse and the increment of velocity from the elliptical transfer orbit at $r = r_{\text{Mars}}$ to the circular orbit of Mars ΔU_{Mars}, the total increment for the whole journey is accomplished in two impulses and may be expressed as

$$\Delta u_{\text{Oberth}} = U_{\text{Oberth}} + \Delta U_{\text{Mars}} \tag{5.82}$$

Based on the treatment outlined, several trajectories are studied. The results are collected in Table 5.2. The trajectories are shown in Fig. 5.7. Details of two impulses for a space vehicle to reach Mars are given in Example 5.5.

Example 5.5

Suppose that we send a space probe from Earth to Mars. When the probe reaches Mars, it will get into a spiral orbit around Mars to make close observations. The

*For Oberth's approach, see Hill, P. G., and Peterson, C. R., *Mechanics and Thermodynamics of Propulsion*, McGraw-Hill, New York, 1983.

Table 5.2 Characteristics of different trajectories

Name of trajectory	Eccentricity, ε	U_{Oberth}, km/s	ΔU_{Mars}, km/s	Time required, days
Hohmann	0.2075	11.6	2.6	256
Ellipse 1	0.2525	11.7	−2.6[a]	175
Ellipse 2	0.3418	12.1	−5.0[a]	135
Parabolic	1	16.7	−16.9[a]	70

[a]Note that the explanation of the case "Ellipse 1" is given in Example 5.5. The detailed expressions for the velocity vectors of the space vehicle and Mars for "Ellipse 2" and "Parabolic trajectories" are given at the end of Example 5.5. To verify these two cases, see the exercises in Problems 5.10 and 5.11.

length of the major axis is chosen as 4.0×10^{11} m for the elliptic trajectory with the center of the sun as the focus. 1) Determine the impulse required for the probe leaving Earth. 2) Determine the required impulse to reduce the velocity of the space probe so that it will have the orbit spiraling down to the surface of Mars. 3) Find the traveling time for the probe from Earth to the Mars circular orbit around the sun.

Solution. 1) Take the radius of the Earth's circular orbit as the r_{min} of the elliptic orbit. It is known that $r_{\text{Earth}} = 1.495 \times 10^{11}$ m. Therefore,

$$\begin{aligned} c &= a - r_{\text{Earth}} \\ &= (2.0 - 1.495) \times 10^{11} = 0.505 \times 10^{11} \qquad \text{(m)} \end{aligned}$$

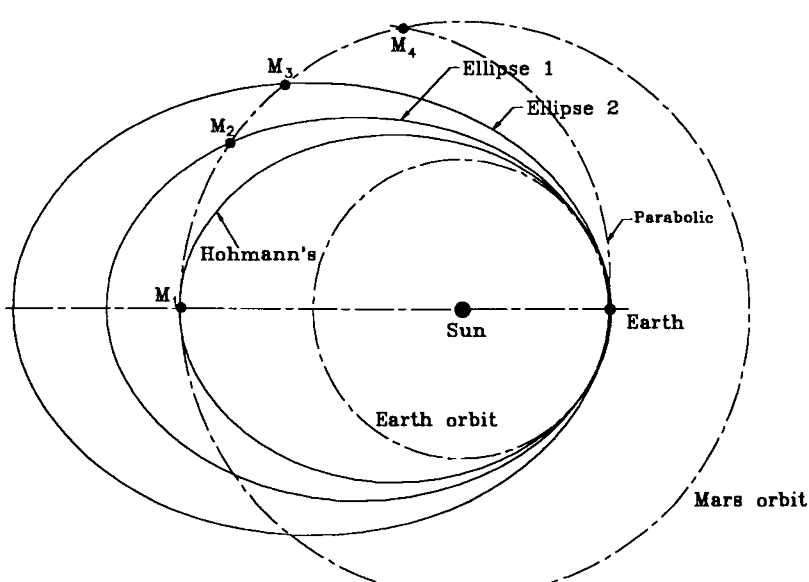

Fig. 5.7 Four different trajectories for Earth–Mars journey.

and

$$\varepsilon = \frac{c}{a} = 0.2525.$$

On the transfer orbit, the velocity of the probe at the surface of Earth is

$$U_T = \sqrt{GM_{\text{sun}}\left(\frac{2}{r_{\text{Earth}}} - \frac{1}{a}\right)}$$

$$= \sqrt{(6.67 \times 10^{-11} \times 1.9866 \times 10^{30})\left(\frac{2}{1.495 \times 10^{11}} - \frac{1}{2 \times 10^{11}}\right)}$$

$$= 3.3343 \times 10^4 \qquad (\text{m/s})$$

The speed of Earth in the circular orbit around the sun is

$$U_e = \sqrt{\frac{GM_{\text{sun}}}{r_{\text{Earth}}}} = 2.98 \times 10^4 \qquad (\text{m/s})$$

and the escape velocity of the space probe leaving the Earth is

$$U_{\text{esc}} = \sqrt{\frac{2GM_e}{R_e}} = 1.118 \times 10^4 \qquad (\text{m/s})$$

where R_e is the radius of the Earth.

To launch the space probe into the transfer orbit directly from the surface of Earth, the required impulse is

$$U_{\text{Oberth}} = \sqrt{U_{\text{esc}}^2 + (U_T - U_e)^2} = 1.1724 \times 10^4 \qquad (\text{m/s})$$

2) To find the required impulse to reduce the velocity of the space probe so that it can spiral down to Mars, we must determine first the intersection point between the elliptic transfer orbit and the Mars circular orbit, then the velocity of the space probe and the relative velocity between the probe and the Mars at that point. From the study of the auxiliary circle of elliptic orbit, we have

$$r = a(1 - \varepsilon \cos\phi)$$

At the intersection point $r = r_{\text{Mars}} = 2.278 \times 10^{11}$ m, we find

$$\phi = 2.15375 \qquad (\text{rad})$$

and

$$\cos\theta = (1/r)(a\cos\phi - c) = -0.7050$$

$$\theta = 2.3532 \qquad (\text{rad})$$

Hence the intersection point is at $r = 2.278 \times 10^{11}$ m and $\theta = 2.3532$ rad. On the transfer orbit with $r = r_{\text{Mars}}$, we have

$$U_T = \sqrt{GM_{\text{sun}}\left(\frac{2}{r_{\text{Mars}}} - \frac{1}{a}\right)} = 2.2395 \times 10^4 \qquad \text{(m/s)}$$

From the orbital equation, we obtain

$$\frac{\mathcal{L}^2}{mk} = \frac{(r^2\dot{\theta})^2}{GM_{\text{sun}}} = r(1 + \varepsilon\cos\theta)$$

$$= 2.278 \times 10^{11}(1 + 0.2525\cos 2.3532) = 1.8725 \times 10^{11} \qquad \text{(m)}$$

$$r\dot{\theta} = 21{,}882 \qquad \text{(m/s)}$$

$$\dot{r} = \sqrt{U_T^2 - (r\dot{\theta})^2} = 4767 \qquad \text{(m/s)}$$

Therefore,

$$\boldsymbol{U}_T = \dot{r}\boldsymbol{e}_r + (r\dot{\theta})\boldsymbol{e}_\theta = 4767\boldsymbol{e}_r + 21{,}882\boldsymbol{e}_\theta \qquad \text{(m/s)}$$

On the other hand, the velocity of Mars on its circular orbit is $24{,}100\boldsymbol{e}_\theta$ m/s Hence, the relative velocity between the space probe and Mars is

$$\boldsymbol{U}_{T-M} = \boldsymbol{U}_T - \boldsymbol{U}_M = 4767\boldsymbol{e}_r - 2218\boldsymbol{e}_\theta \qquad \text{(m/s)}$$

Transform this velocity to an observer on the surface of Mars with the unit vectors denoted by $(\boldsymbol{i}_r, \boldsymbol{i}_\theta)$ on Mars. They are related to the unit vectors in the transfer orbit by $\boldsymbol{i}_r = -\boldsymbol{e}_\theta$, $\boldsymbol{i}_\theta = \boldsymbol{e}_r$. To that observer, he finds that the velocity of the probe at the surface of Mars is

$$v_p = 2218\boldsymbol{i}_r + 4767\boldsymbol{i}_\theta \qquad \text{(m/s)}$$

With this velocity the space probe will have a hyperbolic orbit around Mars. However, if a proper impulse is applied, the probe can stay in the vicinity of Mars. We determine the required reduction of velocity by setting the tangential velocity less than the tangential velocity needed to balance the centrifugal force on a circular orbit and the radial component zero. For a circular orbit of radius of 3500 km, which is slightly greater than the radius of Mars (3332 km), the tangential velocity is

$$V_\theta = \sqrt{\frac{GM_{\text{Mars}}}{R_P}} = \sqrt{\frac{6.67 \times 10^{-11} \times 0.63873 \times 10^{24}}{3{,}500{,}000}} = 3489 \qquad \text{(m/s)}$$

where R_P is the radius of the probe position measured from the center of Mars. From this calculation we determine the required reduction in velocity by choosing $v_p' = 3480\boldsymbol{i}_\theta$ (m/s),

$$v_P' - v_p = -1287\boldsymbol{i}_\theta - 2218\boldsymbol{i}_r$$

and

$$\Delta v = |v'_P - v_P| = \sqrt{1287^2 + 2218^2}$$
$$= 2564 \qquad \text{(m/s)}$$

Therefore, the velocity of the space probe is reduced to the velocity less than the velocity for circular orbit. With this velocity, the space probe will stay in the vicinity of Mars. The radius of the orbit is expected to decrease gradually, spiraling down to the surface of Mars because its centrifugal force is slightly less than the gravitational force for a circular orbit.

3) The traveling time of the space probe from the surface of Earth to the Mars circular orbit around the sun is calculated as follows.

The period for the whole elliptic trajectory is

$$T = 2\pi\sqrt{\frac{a^3}{GM_{\text{sun}}}} = \frac{2\pi(2\times 10^{11})^{1.5}}{\sqrt{1.352\times 10^{20}}} = 565.06 \text{ days}$$

The time required for traveling from $\phi = 0$ to $\phi = 2.15375$ is

$$t = \frac{T}{2\pi}(\phi - \varepsilon\sin\phi) = 174.7 \text{ days}$$

Note that in Table 5.2, the values of ΔU_{Mars} for the cases of ellipse 2 and parabolic trajectories are computed with the considerations of spiraling orbits around Mars as given in this example. Detailed expressions between the velocity vectors of the space vehicle and Mars are given as follows:

$$(\Delta U_{\text{Mars}})_{\text{ell},2} = 8.236\boldsymbol{e}_r - 1.451\boldsymbol{e}_\theta \qquad \text{(km/s)}$$
$$(\Delta U_{\text{Mars}})_{\text{parab}} = 20.01\boldsymbol{e}_r - 3.551\boldsymbol{e}_\theta \qquad \text{(km/s)}$$

Problems

5.1. A single-stage rocket is launched vertically from the surface of the Earth. The velocity and position of the rocket at the burnout are predetermined. Suppose that the mass ratio (m_0/m_b) is also given. Find the required mass flow rate and the exhaust velocity at the nozzle exit to launch such a rocket.

5.2. Compare the terminal payload velocities between a single-stage rocket and a two-stage rocket with same payload ratio of m_L/m_{01}, structure coefficient, and the exhaust velocity. Suppose that the initial mass $m_{01} = 100{,}000$ kg, payload $m_L = 2000$ kg, the structure coefficient $\epsilon = 0.15$, and the exhaust velocity $v_e = 3500$ m/s. Neglect the gravity and the air drag.

5.3. A satellite is launched from the surface of the Earth. At the time of burnout, the satellite is located at altitude of 1000 km with the radial velocity of $v_r = 500$ m/s. Determine the required tangential velocity such that the minimum radius of the orbit is 7000 km.

5.4. A particle moves in an elliptical orbit of major axis $2a$ and minor axis $2b$, with the origin at the center of the ellipse. If the radius vector to the particle sweeps out area at a constant rate as usual, find the law of force in terms of the mass m and period P of the motion. If the minor axis $2b$ approaches zero under the same force law, what kind of motion would result?

5.5. The gravitational potential for the inverse square law of force is $-k/r$. Suppose a small variation δ/r^2 is added to the potential. Find the general orbital equation. Show that, if δ is a constant and $\delta \ll \mathcal{L}^2/2m$, the orbit is given by an ellipse with major axis precessing slowly, having angular velocity of precession given by $\delta/(\mathcal{L}a^2\sqrt{1-\varepsilon^2})$.

5.6. Take the speed of a planet (or satellite) in an elliptical orbit.

(a) Prove that the speed at the point when the planet is at its maximum distance from the major axis is equal to the geometric mean of the maximum and minimum orbital speeds.

(b) Show that the ratio of extreme orbital speeds (at perihelion and aphelion) is $(1+\varepsilon)/(1-\varepsilon)$.

(c) Take the Earth's eccentricity as 0.0167 and that of Halley's comet as 0.967; calculate the ratio in part (b) for each.

5.7. With the use of the Runge–Kutta method, find the trajectory of an electrical propulsion rocket with the initial conditions in dimensionless form $\rho = 1$, $\dot{\rho} = \ddot{\rho} = 0$, and the parameter $\upsilon = 0.001$. Plot the computed results.

5.8. Prove that the solutions obtained from the small perturbation method satisfy the differential equations and the initial conditions for the electrical propulsion rocket.

5.9. A satellite is launched from the surface of the Earth. At the time of burnout, the satellite is located at an altitude of 700 km with velocity of $\upsilon = 1000\boldsymbol{e}_r + 5000\boldsymbol{e}_\theta$ (m/s). Determine the impulse required to increase the velocity in the tangential direction when υ_r is zero, so that the orbit of satellite is circular around the Earth.

5.10. Verify the results of the ellipse 2 trajectory in Table 5.2 for a space vehicle from the surface of Earth to Mars' orbit. The length of major axis $2a$ is 4.5×10^{11} m.

5.11. Verify the results of the parabola trajectory in Table 5.2 for a space vehicle from the surface of Earth to Mars's orbit.

6
Matrices, Tensors, Dyadics, and Rotation Operators

THIS chapter is intended to familiarize students with the mathematical symbols used in technical journals so that they may better understand newly published papers and to provide background for studying motions of rigid bodies in Chapter 7. These mathematical symbols allow many equations to be written in concise forms. Some topics, which are usually covered in a course of applied mathematics, will be introduced with a minimal amount of new physical concepts. To understand the subjects in this chapter better, students should have two courses in calculus and one course in differential equations and, specifically, some basic knowledge of matrix operations (see Appendix D).

Section 6.1 will show the relationship between two orthogonal coordinate systems under rotational motion relative to each other. Matrix notation and operations are introduced. Applications of matrix operations are given in Section 6.2 and in later sections dealing with the study of rotation of a symmetrical top. Section 6.3 introduces Cartesian tensors and dyadics including some basic operations. Applications of these are given in Sections 6.4, 6.5, and 6.6. Rotation operators are described in Section 6.7. The use of the rotation operator can simplify descriptions of complicated rotational motions. Some examples are given to illustrate this point. In general, this chapter provides background for studying the motions of rigid bodies.

6.1 Linear Transformation Matrices

From analytical geometry, we know that when x', y' axes are rotated with respect to the z axis by an angle of θ relative to the x, y axes as shown in Fig. 6.1; the relation of x', y' to x and y can be written as

$$x' = (\cos\theta)x + (\sin\theta)y \tag{6.1}$$

$$y' = (-\sin\theta)x + (\cos\theta)y \tag{6.2}$$

From Eqs. (6.1) and (6.2), we can solve easily for x, y in terms of x' and y' as

$$x = (\cos\theta)x' - (\sin\theta)y' \tag{6.3}$$

$$y = (\sin\theta)x' + (\cos\theta)y' \tag{6.4}$$

Equations (6.1) and (6.2) or (6.3) and (6.4) are examples of a linear transformation from one set of quantities to another. These quantities can be obtained in a different way. Considering a position vector $\boldsymbol{r}$ extending from the origin to the point P, we write

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} = x'\boldsymbol{i}' + y'\boldsymbol{j}'$$

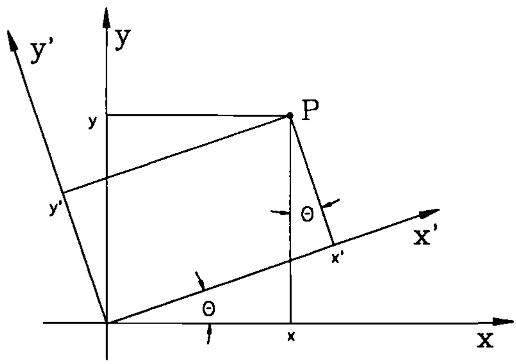

Fig. 6.1 Relation between prime and unprimed systems.

Then $\boldsymbol{r} \cdot \boldsymbol{i}'$ and $\boldsymbol{r} \cdot \boldsymbol{j}'$ will lead to Eqs. (6.1) and (6.2). Similarly, $\boldsymbol{r} \cdot \boldsymbol{i}$ and $\boldsymbol{r} \cdot \boldsymbol{j}$ will give Eqs. (6.3) and (6.4). Extending this technique to a three-dimensional vector, we have

$$x' = \cos(\boldsymbol{i}', \boldsymbol{i})x + \cos(\boldsymbol{i}', \boldsymbol{j})y + \cos(\boldsymbol{i}', \boldsymbol{k})z$$
$$y' = \cos(\boldsymbol{j}', \boldsymbol{i})x + \cos(\boldsymbol{j}', \boldsymbol{j})y + \cos(\boldsymbol{j}', \boldsymbol{k})z$$
$$z' = \cos(\boldsymbol{k}', \boldsymbol{i})x + \cos(\boldsymbol{k}', \boldsymbol{j})y + \cos(\boldsymbol{k}', \boldsymbol{k})z$$

where $\cos(\boldsymbol{i}', \boldsymbol{i})$ is the cosine function of the angle between $\boldsymbol{i}'$ and $\boldsymbol{i}$. To simplify the notation, we let

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z$$
$$a_{11} = \cos(\boldsymbol{i}', \boldsymbol{i}), \qquad a_{12} = \cos(\boldsymbol{i}', \boldsymbol{j}), \qquad a_{13} = \cos(\boldsymbol{i}', \boldsymbol{k})$$
$$a_{21} = \cos(\boldsymbol{j}', \boldsymbol{i}), \qquad a_{22} = \cos(\boldsymbol{j}', \boldsymbol{j}), \qquad a_{23} = \cos(\boldsymbol{j}', \boldsymbol{k})$$
$$a_{31} = \cos(\boldsymbol{k}', \boldsymbol{i}), \qquad a_{32} = \cos(\boldsymbol{k}', \boldsymbol{j}), \qquad a_{33} = \cos(\boldsymbol{k}', \boldsymbol{k})$$

Then we have

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$
$$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

or

$$x_i' = \sum_{j=1}^{3} a_{ij}x_j, \quad i = 1, 2, 3 \tag{6.5}$$

where a_{ij} are the direction cosines for all i and j. Because the magnitude of $\boldsymbol{r}$ does not change from one system to another, clearly

$$\sum_j x_j^2 = \sum_i x_i'^2 \tag{6.6}$$

With the use of Eq. (6.5) in Eq. (6.6), we have

$$\sum_j x_j^2 = \sum_j \left(\sum_k a_{jk}x_k\right)\left(\sum_\ell a_{j\ell}x_\ell\right) = \sum_{k,\ell} x_k x_\ell \left(\sum_j a_{jk}a_{j\ell}\right)$$

The commutative property of addition has been used in the preceding manipulation. For the two sides to agree, we let

$$\sum_j a_{jk}a_{j\ell} = \delta_{k,\ell} \tag{6.7}$$

where $\delta_{k,\ell}$ is the Kronecker delta function with the property

$$\delta_{k,\ell} \equiv \begin{cases} 1 & \text{as } k = \ell \\ 0 & \text{as } k \neq \ell \end{cases}$$

Equation (6.7) is known as the orthogonality condition on the direction cosines, and the transformation Eq. (6.5) consequently is called an orthogonal transformation, which transforms one set of orthogonal coordinates into another set.

The orthogonal transformation can be written in matrix notation as

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X} \tag{6.8}$$

where

$$\boldsymbol{X}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \qquad \boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Let us review some basic operations from matrix algebra. Note that $\boldsymbol{A}$ is a square matrix. If $\boldsymbol{A}_{ij}$ is the cofactor of $\boldsymbol{a}_{ij}$ in the determinant of $\boldsymbol{A}$, then the matrix

$$(\boldsymbol{A}_{ji}) \equiv \text{transpose of } (\boldsymbol{A}_{ij})$$

is called the adjoint of $\boldsymbol{A}$. The reciprocal or inverse of a nonsingular matrix $\boldsymbol{A}$ is the adjoint of $\boldsymbol{A}$ divided by the determinant of $\boldsymbol{A}$. The reciprocal of $\boldsymbol{A}$ is denoted by the symbol $\boldsymbol{A}^{-1}$. Therefore,

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|}\text{Adj}(\boldsymbol{A}) \tag{6.9}$$

Multiplying Eq. (6.8) by $\boldsymbol{A}^{-1}$ leads to

$$\boldsymbol{A}^{-1}\boldsymbol{X}' = \boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{X}$$

or

$$X = A^{-1}X' \tag{6.10}$$

This is known as Cramer's rule. However, because A is formed by direction cosines of an orthogonal transformation, Eq. (6.10) can be simplified further to

$$X = A^T X' \tag{6.11}$$

where A^T = transpose of A, that is,

$$A^{-1} = A^T \tag{6.12}$$

The proof of Eq. (6.12) is given as follows. Based on

$$AA^{-1} = I$$

let $D = A^{-1}$, then

$$(AD)_{ij} = \sum_k a_{ik} \mathrm{d}_{kj} = (I)_{ij} = \delta_{ij}$$

Multiplying the preceding expression of $a_{i\ell}$ and taking summation over i gives

$$\sum_{i,k} a_{i\ell}(a_{ik}\mathrm{d}_{kj}) = \sum_i a_{i\ell}\delta_{i,j} = a_{j\ell}$$

The left-hand side of the preceding equation is

$$\sum_{i,k} a_{i\ell}(a_{ik}\mathrm{d}_{kj}) = \sum_{i,k}(a_{i\ell}a_{ik})\mathrm{d}_{kj} = \sum_k \delta_{\ell,k}\mathrm{d}_{kj} = \mathrm{d}_{\ell j}$$

Therefore,

$$a_{j\ell} = (A)_{j\ell} = (A^T)_{\ell j} = \mathrm{d}_{\ell j} = (A^{-1})_{\ell j}$$

or

$$A^{-1} = A^T$$

When a matrix satisfies the preceding equation, it is called an orthogonal matrix. To illustrate the use of Eq. (6.12), let us consider

$$A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^T = A^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We obtain

$$AA^T = \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Example 6.1

Consider an airplane flying in a horizontal plane and measuring the wind velocity of a hurricane. A coordinate system (x', y', z'), which is attached to the airplane, is the moving system. Another system (x, y, z), which is fixed to earth with x–y plane parallel to the surface of the earth, is the fixed system. To simplify the problem, assume that $x'y'z'$ coordinates coincide with xyz coordinates at the beginning of operation; however, at the instant of measurement the airplane has yawed with respect to the z axis by an angle of θ. The wind velocity is successfully measured by the airplane in the $x'y'z'$ system. What is the velocity in xyz system?

Solution. It is known that $X' = RX$ and $X = R^T X'$. Applying this relationship for the transformation of velocity vector, we have

$$V = R^T V'$$

where

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R^T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v'_x \\ v'_y \\ v'_z \end{pmatrix} = \begin{pmatrix} v'_x\cos\theta - v'_y\sin\theta \\ v'_x\sin\theta + v'_y\cos\theta \\ v'_z \end{pmatrix}$$

6.2 Application of Linear Transformation to Rotation Matrix

From the first course in dynamics, we know that six degrees of freedom are necessary to specify a solid body in motion: x, y, and z for a specific point on the body and θ, ϕ, and ψ for angular displacements of the body relative to a set of fixed axes. To illustrate the application of linear transformation, let us consider a solid body that is rotating without translational motion. Suppose that x, y, z

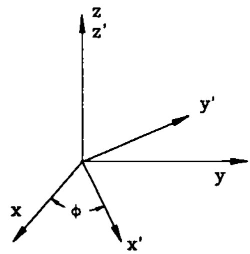

Fig. 6.2 Relative position between x' and x systems.

are fixed coordinates and that the prime system is attached to the rotating body. Consider the rotation in three steps as follows.

1) Let x', y', z' coincide with x, y, z first, and then rotate x', y', z' counterclockwise by angle ϕ about z as shown in Fig. 6.2. The relationship between the prime system and the fixed system is

$$X' = R_1 X$$

where

$$R_1 = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.13}$$

2) Let x'', y'', z'' coincide with x', y', z' first, and then rotate x'', y'', z'' counterclockwise by angle θ about x' as shown in Fig. 6.3. The relation between x'', y'', z'' and x, y, z is

$$X'' = R_2 X' = R_2(R_1 X) = (R_2 R_1)X$$

where

$$R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{6.14}$$

The intersection of the x–y and x''–y'' planes is called the line of nodes.

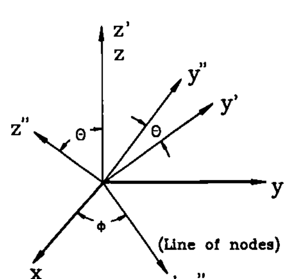

Fig. 6.3 Relative position between X'' and X' systems.

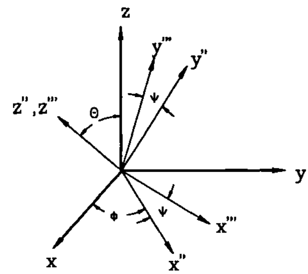

Fig. 6.4 Relative position between X''' and X''.

3) Let x''', y''', z''' coincide with x'', y'', z'' first, and then rotate x''', y''', z''' counterclockwise by angle ψ about z'' as shown in Fig. 6.4. Then the relation between x''', y''', z''' and x, y, z is

$$X''' = R_3 X'' = (R_3 R_2 R_1) X = RX$$

where

$$R_3 = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.15}$$

$$R = R_3 R_2 R_1$$

$$R = \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{pmatrix} \tag{6.16}$$

The angles ϕ, θ, and ψ are known as Euler angles and are used to study the motion of a rotating top in Chapter 7.

6.3 Cartesian Tensors and Dyadics

A tensor is a quantity similar to a vector but with a much broader sense. Tensors can be scalars such as temperature or energy; tensors can be vectors; some tensors can represent stress, strain, or moment of inertia of a solid body. Furthermore, some high rank tensors can be used to express quantities in n-dimensional space with $n > 3$. The knowledge of tensors is essential for the study of general relativity theory. In this section, however, we are going to study only the Cartesian tensor. That means that the axes of the coordinate system, primed or unprimed, are perpendicular to each other.

Cartesian Tensor

A Cartesian tensor T in three-dimensional space is defined as a quantity that transforms according to the rule

$$T'_{lmn,\dots} = \sum_{i,j,k=1}^{3} a_{li}\, a_{mj}\, a_{nk} \cdots T_{ijk,\dots} \tag{6.17}$$

in a rotation from unprimed to primed coordinates where a_{li}s are direction cosines between the axes in unprimed and primed systems. The $T_{ijk,...}$ are called the components of the tensor and are functions of the unprimed coordinates; the $T'_{lmn,...}$ are the corresponding components in the primed system.

The rank of the tensor is defined by the total number of indices. Therefore, T is a zero-rank tensor that is a scalar such as temperature or energy; T_i is a first-rank tensor that is a vector such as velocity, force, and torque, etc.; T_{ij} is a second-rank tensor that represents nine-dimensional quantities in three-dimensional space such as stress and strain. In this chapter most of our attention will be devoted to the second-rank tensor.

A first-rank tensor is simply a vector. The transformation of a vector from unprimed system to primed system is

$$T'_\ell = \sum_i a_{\ell i} T_i$$

Consider that A_i and B_j are two first-rank tensors. Their transformations are

$$A'_\ell = \sum_i a_{\ell i} A_i \qquad \text{and} \qquad B'_\ell = \sum_j a_{\ell j} B_j$$

The dot product of $\boldsymbol{A}'$ and $\boldsymbol{B}'$ is

$$\sum_\ell A'_\ell B'_\ell = \sum_{\ell,i,j} (a_{\ell i} A_i)(a_{\ell j} B_j) = \sum_{i,j} \left(\sum_\ell a_{\ell i} a_{\ell j} \right) A_i B_j$$

$$= \sum_{i,j} (\delta_{i,j}) A_i B_j = \sum_i A_i B_i = \boldsymbol{A} \cdot \boldsymbol{B}$$

In other words, the dot product of any two vectors is invariant under the rotation of the coordinate system or is of zero rank. Therefore, it is also called isotropic tensor.

Second-Rank Tensor

To understand the second-rank tensor, let us consider

$$T_{ij} = x_i x_j$$

The complete expression of all the components of the tensor can be written in the form of matrix as

$$(T_{ij}) = \begin{pmatrix} x_1^2 & x_1 x_2 & x_1 x_3 \\ x_2 x_1 & x_2^2 & x_2 x_3 \\ x_3 x_1 & x_3 x_2 & x_3^2 \end{pmatrix}$$

The transformation of T_{ij} from unprimed coordinate system to primed system is as follows:

$$\sum_{i,j} a_{\ell i} a_{mj} T_{ij} = \left(\sum_i a_{\ell i} x_i \right) \left(\sum_j a_{mj} x_j \right) = x'_\ell x'_m = T'_{\ell m}$$

Therefore $T_{ij} = x_i x_j$ is a second-rank tensor.

It is important to learn the process of contraction. Looking at

$$T'_{\ell m} = \sum_{i,j} a_{\ell i} a_{mj} T_{ij} \tag{6.18}$$

there are six indices in the right-hand side of the equation. Summing over i and j reduces the rank from six to two. This is called contraction. Note also the rank of a tensor must be the same on both sides of the equation. In many books, the summation sign is omitted in the equations. Automatic summation is to be done over a repeated index. This is known as the Einstein summation convention. For the sake of clarity, however, the summation sign is kept throughout this book.

Dyadic

Dyadic is closely related with vectors and second-rank tensors. A pair of vectors written in a definite order, such as ***ij***, is called a dyad, and a linear combination of dyads is known as a dyadic. For example, a second-rank tensor can be written into dyadic form as

$$\overleftrightarrow{T} = T_{11}\boldsymbol{ii} + T_{12}\boldsymbol{ij} + T_{13}\boldsymbol{ik} + T_{21}\boldsymbol{ji} + \cdots \tag{6.19}$$

Similarly,

$$\begin{aligned}\boldsymbol{AB} = {} & A_x B_x \boldsymbol{ii} + A_x B_y \boldsymbol{ij} + A_x B_z \boldsymbol{ik} + A_y B_x \boldsymbol{ji} + A_y B_y \boldsymbol{jj} + A_y B_z \boldsymbol{jk} \\ & + A_z B_x \boldsymbol{ki} + A_z B_y \boldsymbol{kj} + A_z B_z \boldsymbol{kk}\end{aligned}$$

is a dyadic.

Because vectors are used explicitly in the dyadic, many vector operations can be applied to dyadic operations. Let us study some fundamental operations as follows:

$$\begin{aligned}\boldsymbol{C} \cdot (\boldsymbol{AB}) = {} & C_x(A_x B_x \boldsymbol{i} + A_x B_y \boldsymbol{j} + A_x B_z \boldsymbol{k}) + C_y(A_y B_x \boldsymbol{i} + A_y B_y \boldsymbol{j} + A_y B_z \boldsymbol{k}) \\ & + C_z(A_z B_x \boldsymbol{i} + A_z B_y \boldsymbol{j} + A_z B_z \boldsymbol{k}) = (\boldsymbol{C} \cdot \boldsymbol{A})\boldsymbol{B}\end{aligned}$$

The result shows that it is a vector in the direction of $\boldsymbol{B}$. On the other hand, the dot product of $(\boldsymbol{AB})$ with $\boldsymbol{C}$ from the right-hand side is

$$(\boldsymbol{AB}) \cdot \boldsymbol{C} = \boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C})$$

The vector in the result is in the direction of $\boldsymbol{A}$. Therefore

$$\boldsymbol{C} \cdot (\boldsymbol{AB}) \neq (\boldsymbol{AB}) \cdot \boldsymbol{C} \tag{6.20}$$

A unit dyadic is defined as

$$\overleftrightarrow{1} = \boldsymbol{ii} + \boldsymbol{jj} + \boldsymbol{kk}$$

which possesses the property of

$$\overleftrightarrow{1} \cdot \omega = \omega = \omega \cdot \overleftrightarrow{1} \tag{6.21}$$

where ω represents any vector.

Now let us consider the transformation of a dyadic from an unprimed coordinate system to a primed coordinate system. Suppose that there is a relationship between vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ in the unprimed system as

$$\boldsymbol{U} = \overleftrightarrow{T} \cdot \boldsymbol{V} \tag{6.22}$$

where $\overleftrightarrow{T}$ is a dyad. Note that Eq. (6.22) can be written in matrix form as

$$\boldsymbol{U} = \boldsymbol{T}\boldsymbol{V} \tag{6.23}$$

Transform $\boldsymbol{U}$ and $\boldsymbol{V}$ into $\boldsymbol{U}'$ and $\boldsymbol{V}'$ by premultiplying $\overleftrightarrow{A}$ to Eq. (6.22):

$$\boldsymbol{U}' = \overleftrightarrow{A} \cdot \boldsymbol{U} = \overleftrightarrow{A} \cdot \overleftrightarrow{T} \cdot \boldsymbol{V} \tag{6.24}$$

The equivalent operation in the matrix form is

$$\boldsymbol{U}' = \boldsymbol{A}\boldsymbol{U} = \boldsymbol{A}\boldsymbol{T}\boldsymbol{V}$$

However it is known in the matrix operation that

$$\boldsymbol{U}' = \boldsymbol{A}\boldsymbol{T}(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{V} = (\boldsymbol{A}\boldsymbol{T}\boldsymbol{A}^{-1})\boldsymbol{A}\boldsymbol{V} = (\boldsymbol{A}\boldsymbol{T}\boldsymbol{A}^{-1})\boldsymbol{V}' = \boldsymbol{T}'\boldsymbol{V}'$$

which means that

$$\boldsymbol{T}' = \boldsymbol{A}\boldsymbol{T}\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{T}\boldsymbol{A}^{T} \tag{6.25}$$

Applying this matrix manipulation to Eq. (6.24), we find that

$$\boldsymbol{U}' = \overleftrightarrow{A} \cdot \overleftrightarrow{T} \cdot (\overleftrightarrow{A}^{T}\overleftrightarrow{A}) \cdot \boldsymbol{V} = (\overleftrightarrow{A} \cdot \overleftrightarrow{T} \cdot \overleftrightarrow{A}^{T}) \cdot (\overleftrightarrow{A} \cdot \boldsymbol{V}) = \overleftrightarrow{T}' \cdot \boldsymbol{V}' \tag{6.26}$$

Therefore,

$$\overleftrightarrow{T}' = \overleftrightarrow{A} \cdot \overleftrightarrow{T} \cdot \overleftrightarrow{A}^{T} \tag{6.27}$$

where $\overleftrightarrow{A}$ is a dyadic with direction cosines as the elements. Equation (6.27) can be written in tensor notation as

$$T'_{\ell m} = \sum_{i,j} (\boldsymbol{A})_{\ell i} (\boldsymbol{T})_{ij} (\boldsymbol{A}^{-1})_{jm} = \sum_{i,j} a_{\ell i} a_{mj} T_{ij}$$

which agrees with Eq. (6.18).

Example 6.2

Consider that a solid body is under rotational motion. It is rotating about the axes of symmetry. The axes of the coordinate system are chosen so they coincide with the axes of symmetry of the body. Express the relationship between angular momentum and the product of the moment of inertia and angular velocity of the chosen system in dyadic form. Find also the new relationship as the coordinate system is rotated about the z axis by an angle of ϕ.

Solution. According to the given conditions, the components of angular momentum can be written as

$$L_i = I_i\omega_i \quad i = 1, 2, 3$$

In dyadic form

$$\mathbf{L} = (I_1\mathbf{ii} + I_2\mathbf{jj} + I_3\mathbf{kk}) \cdot (\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}) = \overleftrightarrow{I} \cdot \omega \tag{6.28}$$

The angular momentum in the coordinate system rotated about the z axis by angle ϕ is

$$\begin{aligned} L' &= \overleftrightarrow{R}_1 \cdot \mathbf{L} = \overleftrightarrow{R}_1 \cdot \overleftrightarrow{I} \cdot \omega \\ &= \left(\overleftrightarrow{R}_1 \cdot \overleftrightarrow{I} \cdot \overleftrightarrow{R}_1^T\right) \cdot (\overleftrightarrow{R}_1 \cdot \omega) = \overleftrightarrow{I}' \cdot \omega' \end{aligned} \tag{6.29}$$

where

$$\overleftrightarrow{I}' = \overleftrightarrow{R}_1 \cdot \overleftrightarrow{I} \cdot \overleftrightarrow{R}_1^T \tag{6.30}$$

and

$$\omega' = \overleftrightarrow{R}_1 \cdot \omega \tag{6.31}$$

To clarify the preceding operations, let us express the quantities explicitly. We have

$$\begin{aligned} \overleftrightarrow{R}_1 &= \cos\phi\mathbf{i}'\mathbf{i} + \sin\phi\mathbf{i}'\mathbf{j} - \sin\phi\mathbf{j}'\mathbf{i} + \cos\phi\mathbf{j}'\mathbf{j} + \mathbf{k}'\mathbf{k} \\ \mathbf{L}' &= \mathbf{i}'L_1' + \mathbf{j}'L_2' + \mathbf{k}'L_3' \end{aligned} \tag{6.32}$$

$$\begin{aligned} \mathbf{L}' = \overleftrightarrow{R}_1 \cdot \mathbf{L} &= \mathbf{i}'(L_1\cos\phi + L_2\sin\phi) \\ &+ \mathbf{j}'(-L_1\sin\phi + L_2\cos\phi) + \mathbf{k}'L_3 \end{aligned} \tag{6.33}$$

Therefore,

$$\begin{aligned} L_1' &= L_1\cos\phi + L_2\sin\phi \\ L_2' &= -L_1\sin\phi + L_2\cos\phi \\ L_3' &= L_3 \end{aligned}$$

To find $\overleftrightarrow{I}'$, we first write $\overleftrightarrow{R}_1^T$ explicitly as

$$\overleftrightarrow{R}_1^T = \cos\phi\mathbf{ii}' - \sin\phi\mathbf{ij}' + \sin\phi\mathbf{ji}' + \cos\phi\mathbf{jj}' + \mathbf{kk}' \tag{6.34}$$

then we obtain

$$\begin{aligned}\overleftrightarrow{I}' &= \overleftrightarrow{R}_1 \cdot \overleftrightarrow{I} \cdot \overleftrightarrow{R}_1^T \\ &= (\cos\phi \boldsymbol{i}'\boldsymbol{i} + \sin\phi \boldsymbol{i}'\boldsymbol{j} - \sin\phi \boldsymbol{j}'\boldsymbol{i} + \cos\phi \boldsymbol{j}'\boldsymbol{j} + \boldsymbol{k}'\boldsymbol{k}) \\ &\cdot (I_1 \boldsymbol{ii} + I_2 \boldsymbol{jj} + I_3 \boldsymbol{kk}) \cdot (\cos\phi \boldsymbol{ii}' - \sin\phi \boldsymbol{ij}' + \sin\phi \boldsymbol{ji}' + \cos\phi \boldsymbol{jj}' + \boldsymbol{kk}') \\ &= \boldsymbol{i}'\boldsymbol{i}'(I_1\cos^2\phi + I_2\sin^2\phi) + \boldsymbol{i}'\boldsymbol{j}'(-I_1 + I_2)\cos\phi\sin\phi \\ &+ \boldsymbol{j}'\boldsymbol{i}'(-I_1 + I_2)\cos\phi\sin\phi + \boldsymbol{j}'\boldsymbol{j}'(I_1\sin^2\phi + I_2\cos^2\phi) + \boldsymbol{k}'\boldsymbol{k}'I_3 \end{aligned} \tag{6.35}$$

Similar to $\boldsymbol{L}'$, we find

$$\boldsymbol{\omega}' = \overleftrightarrow{R}_1 \cdot \boldsymbol{\omega} = \boldsymbol{i}'(\omega_1\cos\phi + \omega_2\sin\phi) + \boldsymbol{j}'(-\omega_1\sin\phi + \omega_2\cos\phi) + \boldsymbol{k}'\omega_3$$

Through $\boldsymbol{L}' = \overleftrightarrow{I}' \cdot \boldsymbol{\omega}'$ we finally obtain

$$\begin{aligned} L_1' &= L_1\cos\phi + L_2\sin\phi \\ &= (I_1\cos^2\phi + I_2\sin^2\phi)(\omega_1\cos\phi + \omega_2\sin\phi) \\ &+ (-I_1 + I_2)\cos\phi\sin\phi(-\omega_1\sin\phi + \omega_2\cos\phi) \\ &= I_1\omega_1\cos\phi + I_2\omega_2\sin\phi \end{aligned} \tag{6.36a}$$

$$\begin{aligned} L_2' &= -L_1\sin\phi + L_2\cos\phi \\ &= (-I_1 + I_2)\cos\phi\sin\phi(\omega_1\cos\phi + \omega_2\sin\phi) \\ &+ (I_1\sin^2\phi + I_2\cos^2\phi)(-\omega_1\sin\phi + \omega_2\cos\phi) \\ &= -I_1\omega_1\sin\phi + I_2\omega_2\cos\phi \end{aligned} \tag{6.36b}$$

$$L_3' = L_3 \tag{6.36c}$$

The result shows that $\boldsymbol{L}'$ obtained from $\overleftrightarrow{I}' \cdot \boldsymbol{\omega}'$ is the same as that from $\overleftrightarrow{R}_1 \cdot \boldsymbol{L}$ and serves to illustrate the dyadic operation.

6.4 Tensor of Inertia

After having studied the fundamentals of dyadics, we are ready to learn some applications. Consider a rigid body in rotational motion. A set of rectangular coordinates is attached to the body and is rotating relative to a set of fixed space coordinates. The origins of the two systems coincide and are not in relative motion. Therefore, the body position can be specified by three angular coordinates such as Euler angles (Section 6.2). Without losing generality, let us consider that the solid body consists of many point masses and that the position vector of m_i is $\boldsymbol{r}_i$. Therefore, the velocity of point mass m_i is

$$\boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{r}_i$$

and the angular momentum of the body is

$$
\begin{aligned}
\boldsymbol{L} &= \sum_i \boldsymbol{r}_i \times m_i \boldsymbol{v}_i = \sum_i m_i \boldsymbol{r}_i \times (\omega \times \boldsymbol{r}_i) \\
&= \sum_i m_i \left[r_i^2 \omega - \boldsymbol{r}_i (\boldsymbol{r}_i \cdot \omega) \right] \\
&= \sum_i m_i \left(r_i^2 \overleftrightarrow{1} - \boldsymbol{r}_i \boldsymbol{r}_i \right) \cdot \omega = \overleftrightarrow{I}_m \cdot \omega
\end{aligned} \tag{6.37}
$$

where $\overleftrightarrow{I}_m$ is the tensor of inertia expressed in dyadic form and

$$
\overleftrightarrow{I}_m = \sum_i m_i \left(r_i^2 \overleftrightarrow{1} - \boldsymbol{r}_i \boldsymbol{r}_i \right) \tag{6.38}
$$

Expanding Eq. (6.37), we have

$$
\begin{aligned}
L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\
L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\
L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z
\end{aligned} \tag{6.39}
$$

where

$$
\begin{aligned}
I_{xx} &= \sum_i m_i \left(r_i^2 - x_i^2 \right), & I_{xy} &= -\sum_i m_i x_i y_i, & I_{xz} &= -\sum_i m_i x_i z_i \\
I_{yx} &= -\sum_i m_i y_i x_i, & I_{yy} &= \sum_i m_i \left(r_i^2 - y_i^2 \right), & I_{yz} &= -\sum_i m_i y_i z_i \\
I_{zx} &= -\sum_i m_i z_i x_i, & I_{yz} &= -\sum_i m_i z_i y_i, & I_{zz} &= \sum_i m_i \left(r_i^2 - z_i^2 \right)
\end{aligned}
$$

In the preceding expressions, I_{ii} elements are called the moment of inertia and $I_{ij} (i \neq j)$ are the products of inertia.

Now let us relate the inertia tensor to the moment of inertia with respect to the rotational axis of the body. Let $\boldsymbol{n}$ be a unit vector along ω or

$$
\omega = \omega \boldsymbol{n}
$$

The moment of inertia with respect to the rotational axis is simply

$$
\begin{aligned}
I_m &= \boldsymbol{n} \cdot \overleftrightarrow{I}_m \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \sum_i m_i \left(r_i^2 \overleftrightarrow{1} - \boldsymbol{r}_i \boldsymbol{r}_i \right) \cdot \boldsymbol{n} \\
&= \sum_i m_i \left[r_i^2 - (\boldsymbol{r}_i \cdot \boldsymbol{n})^2 \right]
\end{aligned} \tag{6.40}
$$

Through the use of Eq. (6.40), the kinetic energy of the solid body rotating with velocity ω can be expressed in a familiar form as

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i = \frac{1}{2}\sum_i m_i \boldsymbol{v}_i \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_i) \\ &= \frac{1}{2}\sum_i m_i \boldsymbol{r}_i \cdot (\boldsymbol{v}_i \times \boldsymbol{\omega}) = \frac{1}{2}\sum_i m_i \boldsymbol{\omega} \cdot (\boldsymbol{r}_i \times \boldsymbol{v}_i) \\ &= \frac{1}{2}\boldsymbol{\omega} \cdot \left[\sum_i m_i (\boldsymbol{r}_i \times \boldsymbol{v}_i)\right] = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{L} \\ &= \frac{1}{2}\boldsymbol{\omega} \cdot \overleftrightarrow{I}_m \cdot \boldsymbol{\omega} = \frac{1}{2}\omega^2 \boldsymbol{n} \cdot \overleftrightarrow{I}_m \cdot \boldsymbol{n} = \frac{1}{2} I_m \omega^2 \end{aligned}$$

With the definition of inertia tensor given in Eq. (6.38), the generalized parallel axis theorem can be derived easily as follows. Consider

$$\boldsymbol{r}_i = \boldsymbol{r}_i' + \boldsymbol{R}$$

as shown in Fig. 6.5.

The inertia tensor with respect to XYZ coordinates $\overleftrightarrow{I}_0$ then can be expressed as

$$\begin{aligned} \overleftrightarrow{I}_0 &= \sum_i m_i \left(r_i^2 \overleftrightarrow{1} - \boldsymbol{r}_i \boldsymbol{r}_i\right) \\ &= \sum_i m_i [(\boldsymbol{r}_i' + \boldsymbol{R}) \cdot (\boldsymbol{r}_i' + \boldsymbol{R}) \overleftrightarrow{1} - (\boldsymbol{r}_i' + \boldsymbol{R})(\boldsymbol{r}_i' + \boldsymbol{R})] \\ &= \sum_i m_i [\boldsymbol{r}_i' \cdot \boldsymbol{r}_i' \overleftrightarrow{1} - \boldsymbol{r}_i' \boldsymbol{r}_i'] + \sum_i m_i [\boldsymbol{R} \cdot \boldsymbol{R} \overleftrightarrow{1} - \boldsymbol{R}\boldsymbol{R}] \\ &\quad + 2\left[\boldsymbol{R} \cdot \left(\sum_i m_i \boldsymbol{r}_i'\right)\right] \overleftrightarrow{1} - \boldsymbol{R}\left(\sum_i m_i \boldsymbol{r}_i'\right) - \left(\sum_i m_i \boldsymbol{r}_i'\right)\boldsymbol{R} \end{aligned}$$

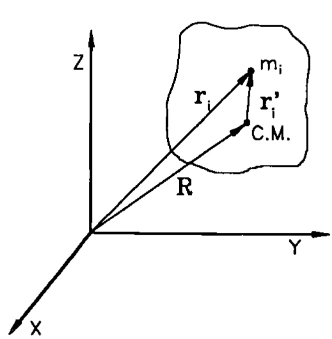

Fig. 6.5 General position of a body in *XYZ* coordinates.

Because the origin of the primed system is chosen at the center of mass,

$$\sum_i m_i \boldsymbol{r}_i' = 0$$

$$\overleftrightarrow{I}_0 = \sum_i m_i \left[r_i'^2 \overleftrightarrow{1} - \boldsymbol{r}_i' \boldsymbol{r}_i' \right] + M[R^2 \overleftrightarrow{1} - \boldsymbol{RR}]$$

$$= \overleftrightarrow{I}_c + M(R^2 \overleftrightarrow{1} - \boldsymbol{RR}) \tag{6.41}$$

where $\overleftrightarrow{I}_c$ is the inertia tensor of the solid body with respect to the primed axes with the origin at the center of mass. Equation (6.41) is known as the generalized parallel axis theorem.

To illustrate the generalized parallel axis theorem, let us consider a case where the center of mass of the body is at distance x on the x axis:

$$\boldsymbol{R} = x\boldsymbol{i}, \qquad R^2 = x^2, \qquad \boldsymbol{RR} = \boldsymbol{ii}x^2$$

$$M(R^2 \overleftrightarrow{1} - \boldsymbol{RR}) = M(x^2 \boldsymbol{jj} + x^2 \boldsymbol{kk})$$

$$\overleftrightarrow{I}_0 = \overleftrightarrow{I}_c + Mx^2(\boldsymbol{jj} + \boldsymbol{kk})$$

Writing the components of the moment of inertia in detail, we have

$$(\overleftrightarrow{I}_0)_{22} = (\overleftrightarrow{I}_c)_{22} + Mx^2 \tag{6.42a}$$

$$(\overleftrightarrow{I}_0)_{33} = (\overleftrightarrow{I}_c)_{33} + Mx^2 \tag{6.42b}$$

$$(\overleftrightarrow{I}_0)_{11} = (\overleftrightarrow{I}_c)_{11} \tag{6.42c}$$

$$(\overleftrightarrow{I}_0)_{ij} = (\overleftrightarrow{I}_c)_{ij} \qquad i \neq j \tag{6.42d}$$

In Eqs. (6.42a) and (6.42b), the difference between $\overleftrightarrow{I}_0$ and $\overleftrightarrow{I}_c$ is Mx^2 in $\boldsymbol{jj}$ and $\boldsymbol{kk}$ because the y' and z' axes are moved by x; however, because x, x' coincide, $(\overleftrightarrow{I}_0)_{11} = (\overleftrightarrow{I}_c)_{11}$. The results given in Eqs. (6.42a–6.42d) agree with the parallel axis theorem written with nine separate equations as given in the first course of dynamics.

6.5 Principal Stresses and Axes in a Three-Dimensional Solid

We have studied the fundamentals of matrices and tensors in Sections 6.1 and 6.3. Now let us apply them to determine the principal stresses in a solid. When forces and torques are applied to a three-dimensional homogeneous solid, three-dimensional stresses are set in the solid. As shown in Fig. 6.6, these stresses have nine components σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, and τ_{zy}. Because these stresses are in equilibrium, the summation of moments with respect to each axis must be equal to zero, and the results show that $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, and $\tau_{yz} = \tau_{zy}$. The state of the stresses can be written in matrix form as

$$\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \tag{6.43}$$

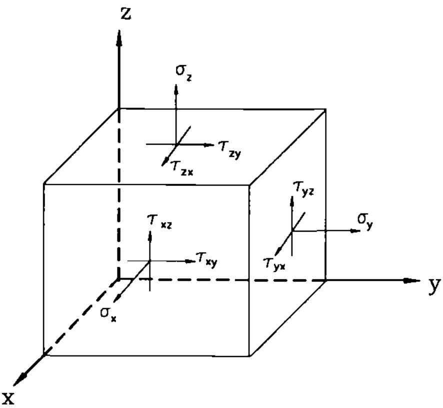

Fig. 6.6 Three-dimensional stress on a solid cube.

in which σ_i is the component of normal stresses and τ_{ij} is the component of shear stresses. The σ is a symmetric matrix. Figure 6.7 shows a tetrahedron formed by drawing three planes normal to the coordinate axes and a fourth plane with a directed normal $\boldsymbol{n}$ at a distance h from the point P that is at the origin. In the limit, as $h \to 0$, the tetrahedron will become of infinitesimal order with sides $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$, and the inclined plane approaches P.

To find the principal stresses in a solid, we assume that the stress acting on the inclined plane is only a normal stress σ_n. The components of σ_n are σ_{nx}, σ_{ny}, and σ_{nz}. In the limit, as $h \to 0$, the equilibrium of all forces in the x direction requires

$$-\tfrac{1}{2}\tau_{yx}\mathrm{d}x\,\mathrm{d}z - \tfrac{1}{2}\tau_{zx}\mathrm{d}x\,\mathrm{d}y - \tfrac{1}{2}\sigma_x\mathrm{d}y\,\mathrm{d}z + \sigma_{nx}\mathrm{d}A = 0$$

where $\mathrm{d}A$ is the area of the inclined plane. Note that

$$\tfrac{1}{2}\mathrm{d}y\,\mathrm{d}z = \mathrm{d}A\boldsymbol{n}\cdot\boldsymbol{i} = \mathrm{d}Aa_{nx}$$

$$\tfrac{1}{2}\mathrm{d}x\,\mathrm{d}z = \mathrm{d}A\boldsymbol{n}\cdot\boldsymbol{j} = \mathrm{d}Aa_{ny}$$

$$\tfrac{1}{2}\mathrm{d}x\,\mathrm{d}y = \mathrm{d}A\boldsymbol{n}\cdot\boldsymbol{k} = \mathrm{d}Aa_{nz}$$

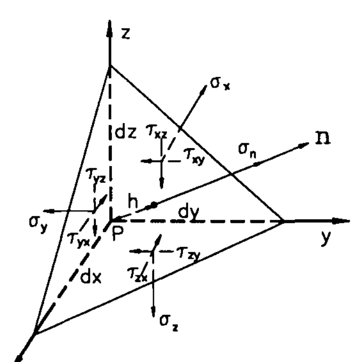

Fig. 6.7 Three-dimensional stress on a tetrahedron.

and

$$\boldsymbol{n} = a_{nx}\boldsymbol{i} + a_{ny}\boldsymbol{j} + a_{nz}\boldsymbol{k}$$

Hence,

$$\sigma_{nx} = \sigma_x a_{nx} + \tau_{yx} a_{ny} + \tau_{zx} a_{nz} \tag{6.44}$$

Similarly, we can find

$$\sigma_{ny} = \tau_{xy} a_{nx} + \sigma_y a_{ny} + \tau_{zy} a_{nz} \tag{6.45}$$

$$\sigma_{nz} = \tau_{zx} a_{nx} + \tau_{yz} a_{ny} + \sigma_z a_{nz} \tag{6.46}$$

On the other hand,

$$\begin{aligned} \sigma_{nx} &= \sigma_n(\boldsymbol{n}\cdot\boldsymbol{i}) = \sigma_n a_{nx} \\ \sigma_{ny} &= \sigma_n a_{ny} \\ \sigma_{nz} &= \sigma_n a_{nz} \end{aligned} \tag{6.47}$$

Therefore, we find that the equations for the balance of forces are

$$\begin{aligned} \sigma_x a_{nx} + \tau_{yx} a_{ny} + \tau_{zx} a_{nz} &= \sigma_n a_{nx} \\ \tau_{xy} a_{nx} + \sigma_y a_{ny} + \tau_{zy} a_{nz} &= \sigma_n a_{ny} \\ \tau_{zx} a_{nx} + \tau_{yz} a_{ny} + \sigma_z a_{nz} &= \sigma_n a_{nz} \end{aligned} \tag{6.48}$$

Rewriting Eq. (6.48) in matrix form, we have

$$\boldsymbol{\sigma x} = \sigma_n \boldsymbol{x} \tag{6.49}$$

where

$$\boldsymbol{x} = \begin{pmatrix} a_{nx} \\ a_{ny} \\ a_{nz} \end{pmatrix} \tag{6.50}$$

From the formulation given, we will determine a_{nx}, a_{ny}, a_{nz}, and σ_n. In addition to the three equations in (6.48), we have

$$\boldsymbol{n}\cdot\boldsymbol{n} = 1 = a_{nx}^2 + a_{ny}^2 + a_{nz}^2 \tag{6.51}$$

Therefore, we have four equations to determine four unknowns.

Rearrange Eq. (6.49) as

$$\boldsymbol{\sigma x} = \sigma_n \mathbf{1}\boldsymbol{x}$$

or

$$(\sigma - \sigma_n \mathbf{1})\boldsymbol{x} = 0$$

Because $\boldsymbol{x}$ cannot be zero, the determinant of the coefficients must vanish, i.e.,

$$|\sigma - \sigma_n \mathbf{1}| = 0$$

Expanding the determinant, we find the functional relationship, called the characteristic equation,

$$\phi(\sigma_n) = \sigma_n^3 - I_1\sigma_n^2 + I_2\sigma_n - I_3 = 0 \tag{6.52}$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2$$

$$I_3 = \sigma_x\sigma_y\sigma_z - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 + 2\tau_{xy}\tau_{xz}\tau_{yz}$$

The three roots of Eq. (6.52), say σ_1, σ_2, and σ_3, are called the principal stresses. Once the principal stresses have been obtained, the direction cosines of the normals of the planes can be found from Eqs. (6.48) and (6.51). The normals are known as principal axes. To illustrate the procedure in detail, let us study the following example.

Example 6.3

Given a stress matrix,

$$\sigma = \begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix} \qquad \text{(MPa)}$$

find the principal stresses and the direction cosines of the principal axes.

Solution. The characteristic equation is

$$\begin{vmatrix} 7-\sigma_n & -2 & 0 \\ -2 & 6-\sigma_n & -2 \\ 0 & -2 & 5-\sigma_n \end{vmatrix} = -\sigma_n^3 + 18\sigma_n^2 - 99\sigma_n + 162 = 0$$

The three roots are

$$\sigma_1 = 3\text{ MPa}, \qquad \sigma_2 = 6\text{ MPa}, \qquad \sigma_3 = 9\text{ MPa}$$

To find the direction cosines, let us use Eq. (6.48) in explicit form

$$(7 - \sigma_n)a_{nx} - 2a_{ny} = 0$$

$$-2a_{nx} + (6 - \sigma_n)a_{ny} - 2a_{nz} = 0$$

$$-2a_{ny} + (5 - \sigma_n)a_{nz} = 0$$

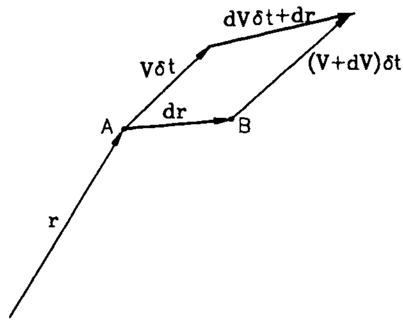

Fig. 6.8 Relative position between *A* and *B*.

For the first root $\sigma_1 = 3$ MPa,

$$4a_{nx} - 2a_{ny} = 0$$
$$-2a_{nx} + 3a_{ny} - 2a_{nz} = 0$$
$$-2a_{ny} + 2a_{nz} = 0$$

In the preceding three equations, only two equations are independent because the determinant of the coefficients is zero. However, we have

$$a_{nx}^2 + a_{ny}^2 + a_{nz}^2 = 1$$

and find

$$a_{nx} = \tfrac{1}{3}, \qquad a_{ny} = a_{nz} = \tfrac{2}{3}$$

Hence

$$\boldsymbol{n}_1 = \tfrac{1}{3}\boldsymbol{i} + \tfrac{2}{3}\boldsymbol{j} + \tfrac{2}{3}\boldsymbol{k} \qquad \text{for } \sigma_1 = 3 \text{ MPa}$$

Similarly, we find

$$\boldsymbol{n}_2 = \tfrac{2}{3}\boldsymbol{i} + \tfrac{1}{3}\boldsymbol{j} - \tfrac{2}{3}\boldsymbol{k} \qquad \text{for } \sigma_2 = 6 \text{ MPa}$$
$$\boldsymbol{n}_3 = -\tfrac{2}{3}\boldsymbol{i} + \tfrac{2}{3}\boldsymbol{j} - \tfrac{1}{3}\boldsymbol{k} \qquad \text{for } \sigma_3 = 9 \text{ MPa}$$

Note that the three axes are perpendicular to each other. It must be pointed out here that the technique illustrated here for principal stresses can be used also for finding principal strains in homogeneous materials and principal moments of inertia for solid bodies.

6.6 Viscous Stress in Newtonian Fluid

Suppose that a point A is located in a Newtonian fluid and is specified by the position vector $\boldsymbol{r}$ as shown in Fig. 6.8. The term *Newtonian fluid* implies the following postulates.

1) The fluid is continuous, and its stress tensor τ_{ij} is a linear function of the rates of strains.

2) The fluid is isotropic, i.e., its properties are independent of direction, and therefore the deformation law is independent of the coordinate axes in which it is expressed.

3) When the fluid is at rest, the deformation law must reduce to the hydrostatic pressure condition, $\tau_{ij} = -p\delta_{i,j}$.

Consider that A is moving with velocity $\boldsymbol{V}$. In the vicinity of A, there is point B that is moving with velocity $\boldsymbol{V} + \mathrm{d}\boldsymbol{V}$. The velocity $\boldsymbol{V}$ and the change of velocity are written as

$$\boldsymbol{V} = V_1\boldsymbol{i} + V_2\boldsymbol{j} + V_3\boldsymbol{k}$$

$$\mathrm{d}\boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial x}\mathrm{d}x + \frac{\partial \boldsymbol{V}}{\partial y}\mathrm{d}y + \frac{\partial \boldsymbol{V}}{\partial z}\mathrm{d}z = \mathrm{d}\boldsymbol{r} \cdot \nabla\boldsymbol{V}$$

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r} = \nabla\boldsymbol{V} \cdot \boldsymbol{n}$$

where $\boldsymbol{n}$ is the unit vector in the direction of $\mathrm{d}\boldsymbol{r}$:

$$\nabla\boldsymbol{V} = \frac{\partial V_1}{\partial x}\boldsymbol{ii} + \frac{\partial V_2}{\partial x}\boldsymbol{ij} + \frac{\partial V_3}{\partial x}\boldsymbol{ik} + \frac{\partial V_1}{\partial y}\boldsymbol{ji} + \frac{\partial V_2}{\partial y}\boldsymbol{jj} + \frac{\partial V_3}{\partial y}\boldsymbol{jk}$$
$$+ \frac{\partial V_1}{\partial z}\boldsymbol{ki} + \frac{\partial V_2}{\partial z}\boldsymbol{kj} + \frac{\partial V_3}{\partial z}\boldsymbol{kk}$$

$\nabla\boldsymbol{V}$ is called the strain rate dyadic. Note that $\nabla\boldsymbol{V}$ denotes strain as a function of time. Now let us define

$$\epsilon_{11} \equiv \frac{\partial V_1}{\partial x}, \qquad \epsilon_{22} \equiv \frac{\partial V_2}{\partial y}, \qquad \epsilon_{33} \equiv \frac{\partial V_3}{\partial z}$$

$$\epsilon_{12} \equiv \frac{1}{2}\left(\frac{\partial V_1}{\partial y} + \frac{\partial V_2}{\partial x}\right), \quad \epsilon_{13} \equiv \frac{1}{2}\left(\frac{\partial V_3}{\partial x} + \frac{\partial V_1}{\partial z}\right), \quad \epsilon_{23} \equiv \frac{1}{2}\left(\frac{\partial V_2}{\partial z} + \frac{\partial V_3}{\partial y}\right)$$

$$h_1 \equiv \frac{1}{2}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right), \quad h_2 \equiv \frac{1}{2}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right), \quad h_3 \equiv \frac{1}{2}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

The strain rate dyadic then becomes

$$\nabla\boldsymbol{V} = \overleftrightarrow{\epsilon} + \overleftrightarrow{\Omega} \tag{6.53}$$

with

$$\overleftrightarrow{\epsilon} = \epsilon_{11}\boldsymbol{ii} + \epsilon_{12}\boldsymbol{ij} + \epsilon_{13}\boldsymbol{ik} + \epsilon_{12}\boldsymbol{ji} + \epsilon_{22}\boldsymbol{jj} + \epsilon_{23}\boldsymbol{jk}$$
$$+ \epsilon_{13}\boldsymbol{ki} + \epsilon_{23}\boldsymbol{kj} + \epsilon_{33}\boldsymbol{kk} \tag{6.54}$$

and

$$\overleftrightarrow{\Omega} = h_3\boldsymbol{ij} - h_2\boldsymbol{ik} - h_3\boldsymbol{ji} + h_1\boldsymbol{jk} + h_2\boldsymbol{ki} - h_1\boldsymbol{kj} \tag{6.55}$$

Note that $\overleftrightarrow{\epsilon}$ is a symmetric dyadic and is called the pure strain rate dyadic, and $\overleftrightarrow{\Omega}$ is an antisymmetric dyadic and is the rotation dyadic.

To find the expression for viscous stress, let us consider a general stress dyadic:

$$\overleftrightarrow{\tau} = \tau_{11}\boldsymbol{ii} + \tau_{12}\boldsymbol{ij} + \tau_{13}\boldsymbol{ik} + \tau_{12}\boldsymbol{ji} + \tau_{22}\boldsymbol{jj} + \tau_{23}\boldsymbol{jk} + \tau_{13}\boldsymbol{ki} + \tau_{23}\boldsymbol{kj} + \tau_{33}\boldsymbol{kk} \tag{6.56}$$

Through the rotation of coordinate axes, we can find the principal stresses and also the principal axes. Along these principal axes, considered as the primed system, we have

$$\overleftrightarrow{\tau}' = \tau'_{11}\boldsymbol{i'i'} + \tau'_{22}\boldsymbol{j'j'} + \tau'_{33}\boldsymbol{k'k'} \tag{6.57}$$

$$(\nabla \boldsymbol{V})' = \epsilon'_{11}\boldsymbol{i'i'} + \epsilon'_{22}\boldsymbol{j'j'} + \epsilon'_{33}\boldsymbol{k'k'} \tag{6.58}$$

The relationship between the viscous stress and the rate of strain along the x' axis may be expressed as

$$\begin{aligned}
\tau'_{11} &= -p + c_1\epsilon'_{11} + c_2\epsilon'_{22} + c_2\epsilon'_{33} \\
&= -p + (c_1 - c_2)\epsilon'_{11} + c_2(\epsilon'_{11} + \epsilon'_{22} + \epsilon'_{33}) \\
&= -p + (c_1 - c_2)\frac{\partial V'_1}{\partial x'} + c_2(\nabla \cdot \boldsymbol{V}) \\
&= -p + (c_1 - c_2)\epsilon'_{11} + c_2(\nabla \cdot \boldsymbol{V})
\end{aligned} \tag{6.59}$$

Without losing generality, let $c_2 \equiv k - \frac{2}{3}\mu$ and $c_1 \equiv c_2 + 2\mu$ in which k and μ are to be determined. Hence

$$\tau'_{11} = -p + \left(k - \tfrac{2}{3}\mu\right)(\nabla \cdot \boldsymbol{V}) + 2\mu\epsilon'_{11} \tag{6.60}$$

To identify the constants, let us consider first

$$\tau'_{11} = \tau'_{22} = \tau'_{33}$$

$$\epsilon'_{11} = \epsilon'_{22} = \epsilon'_{33} = (\nabla \cdot \boldsymbol{V})/3$$

From Eqs. (6.57) and (6.60) we find

$$\begin{aligned}
\overleftrightarrow{\tau}' &= \left(k - \tfrac{2}{3}\mu\right)(\nabla \cdot \boldsymbol{V})\overleftrightarrow{1} + 2\mu\left[\tfrac{1}{3}(\nabla \cdot \boldsymbol{V})\right]\overleftrightarrow{1} - p\overleftrightarrow{1} \\
&= k(\nabla \cdot \boldsymbol{V})\overleftrightarrow{1} - p\overleftrightarrow{1} = \tau'_{11}\overleftrightarrow{1}
\end{aligned}$$

Hence

$$k = \frac{\tau'_{11} + P}{(\nabla \cdot \boldsymbol{V})} \tag{6.61}$$

Because $(\operatorname{div} V)$ means the change of volume per unit volume, k is known as the coefficient of bulk viscosity. Now let us rotate the axes back to the unprimed coordinate system and consider the viscous stress in a general form as

$$\overleftrightarrow{\tau} = \left[-p + \left(k - \tfrac{2}{3}\mu\right)\nabla \cdot V\right]\overleftrightarrow{1} + 2\mu\overleftrightarrow{\epsilon} \tag{6.62}$$

Note that $\overleftrightarrow{\epsilon}$ is the only term affected by the rotation of coordinate axes. Because Eq. (6.62) is always true for all possible conditions, let us apply the equation to a case such that

$$V_1 = V_1(y), \qquad V_2 = V_3 = 0$$

Then

$$\begin{aligned} \epsilon_{12} &= \frac{1}{2}\frac{\partial V_1}{\partial y} = \frac{1}{2}\frac{\mathrm{d}V_1}{\mathrm{d}y} \\ \tau_{12} &= 2\mu\epsilon_{12} = \mu\frac{\mathrm{d}V_1}{\mathrm{d}y} \end{aligned} \tag{6.63}$$

where μ is known as the coefficient of viscosity. Therefore, Eq. (6.62) is the expression for the viscous stress in Newtonian fluid with k the coefficient of bulk viscosity and μ the coefficient of viscosity.

6.7 Rotation Operators

Earlier, in Chapter 3, we studied the collision of missiles in midair. In Example 3.1, the delay time after the first missile launch is given as 60 s. This interval includes the time to rotate the launching equipment to a proper angle. Certainly this operation could be done by using Euler angles, but that approach takes too much time. With the operation given in this section, we will find that the operation is simplified and saves time.

Consider that a position vector $\boldsymbol{r}$ is rotated with respect to vector $\boldsymbol{n}$ by angle β to $\boldsymbol{r}'$. The angle β is measured in a plane perpendicular to $\boldsymbol{n}$, containing the ends of vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ in that plane as shown in Fig. 6.9. Let $\boldsymbol{a}$ be a vector with the

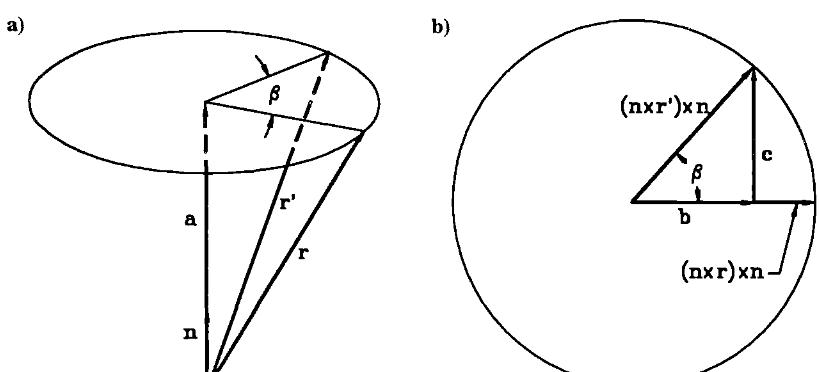

Fig. 6.9 Rotation of $\boldsymbol{r}$ about $\boldsymbol{n}$.

direction of $\boldsymbol{n}$ and the magnitude of the component of $\boldsymbol{r}$ along $\boldsymbol{n}$, so that

$$\boldsymbol{a} = \boldsymbol{n}(\boldsymbol{r} \cdot \boldsymbol{n})$$

Let $\boldsymbol{b}$ and $\boldsymbol{c}$ be vectors in the circular plane, which is the top view of Fig. 6.9a looking down directly along $-\boldsymbol{n}$. Hence

$$\boldsymbol{r}' = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$$

The radius of the circle is $r \sin\theta$ or

$$|\boldsymbol{n} \times \boldsymbol{r}| = |(\boldsymbol{n} \times \boldsymbol{r}) \times \boldsymbol{n}| = |(\boldsymbol{n} \times \boldsymbol{r}') \times \boldsymbol{n}|$$

The vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are

$$\boldsymbol{b} = [(\boldsymbol{n} \times \boldsymbol{r}) \times \boldsymbol{n}] \cos\beta$$

$$\boldsymbol{c} = (\boldsymbol{n} \times \boldsymbol{r}) \sin\beta$$

Finally we have

$$\begin{aligned} \boldsymbol{r}' &= \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{r}) + \cos\beta(\boldsymbol{n} \times \boldsymbol{r}) \times \boldsymbol{n} + \sin\beta(\boldsymbol{n} \times \boldsymbol{r}) \\ &= \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{r}) + [-\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{r}) + \boldsymbol{r}(\boldsymbol{n} \cdot \boldsymbol{n})] \cos\beta + (\boldsymbol{n} \times \boldsymbol{r}) \sin\beta \\ &= (1 - \cos\beta)\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{r}) + \cos\beta \boldsymbol{r} + \sin\beta(\boldsymbol{n} \times \boldsymbol{r}) \end{aligned} \tag{6.64}$$

By defining a rotation operator as

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) = (1 - \cos\beta)\boldsymbol{n}\boldsymbol{n} + \cos\beta \overleftrightarrow{1} + \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) \tag{6.65}$$

we obtain

$$\boldsymbol{r}' = \overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \boldsymbol{r} \tag{6.66}$$

Note that $\boldsymbol{r}'$ is the vector $\boldsymbol{r}$ rotated about $\boldsymbol{n}$ by angle of β. The operator $\overleftrightarrow{R}$ is a function of $\boldsymbol{n}$ and β and is independent of coordinates. Note also that the operator $\overleftrightarrow{R}$ was first introduced by J. W. Gibbs in 1901* and has been further developed by C. Leubner and E. N. Moore.

When

$$\boldsymbol{n} = \boldsymbol{k}$$

$$\overleftrightarrow{R}(\boldsymbol{k}, \beta) = \boldsymbol{k}\boldsymbol{k} + \cos\beta(\boldsymbol{i}\boldsymbol{i} + \boldsymbol{j}\boldsymbol{j}) + \sin\beta(\boldsymbol{j}\boldsymbol{i} - \boldsymbol{i}\boldsymbol{j})$$

Properties of the Operator $\overleftrightarrow{R}(n, \beta)$

1) As $\beta = 0$,

$$\overleftrightarrow{R}(\boldsymbol{n}, 0) = \overleftrightarrow{1} \tag{6.67}$$

*Gibbs, J. W., *Vector Analysis*, Scribner, New York, 1901, Chap. 6.

2) When $\boldsymbol{n}$ is rotated about $\boldsymbol{n}$ itself, $\boldsymbol{n}'$ is $\boldsymbol{n}$ or

$$\overleftrightarrow{R}(\boldsymbol{n},\beta)\cdot\boldsymbol{n}=\boldsymbol{n} \tag{6.68}$$

3) Two consecutive rotations about the same axis $\boldsymbol{n}$ by angles of α and β will expect a result of

$$\overleftrightarrow{R}(\boldsymbol{n},\alpha)\cdot\overleftrightarrow{R}(\boldsymbol{n},\beta)=\overleftrightarrow{R}(\boldsymbol{n},\alpha+\beta) \tag{6.69}$$

The preceding equation, however, requires a mathematical proof, which is given as follows.

It is easily verified that

$$\boldsymbol{A}\cdot(\boldsymbol{n}\times\overleftrightarrow{1})=\boldsymbol{A}\times\boldsymbol{n} \tag{6.70}$$

or

$$(\boldsymbol{n}\times\overleftrightarrow{1})\cdot\boldsymbol{A}=\boldsymbol{n}\times\boldsymbol{A}$$

also

$$(\boldsymbol{n}\times\overleftrightarrow{1})\cdot(\boldsymbol{n}\times\overleftrightarrow{1})=\boldsymbol{nn}-\overleftrightarrow{1} \tag{6.71}$$

and

$$\boldsymbol{nn}\cdot(\boldsymbol{n}\times\overleftrightarrow{1})=\boldsymbol{nn}\times\boldsymbol{n}=0 \tag{6.72}$$

Using Eqs. (6.70–6.72), we have

$$\begin{aligned}
\overleftrightarrow{R}(\boldsymbol{n},\alpha)\cdot\overleftrightarrow{R}(\boldsymbol{n},\beta) &= [(1-\cos\alpha)\boldsymbol{nn}+\cos\alpha\overleftrightarrow{1}+\sin\alpha(\boldsymbol{n}\times\overleftrightarrow{1})] \\
&\cdot[(1-\cos\beta)\boldsymbol{nn}+\cos\beta\overleftrightarrow{1}+\sin\beta(\boldsymbol{n}\times\overleftrightarrow{1})] \\
&= (1-\cos\alpha)(1-\cos\beta)\boldsymbol{nn}+(1-\cos\beta)\cos\alpha\boldsymbol{nn} \\
&+ \sin\alpha(1-\cos\beta)[(\boldsymbol{n}\times\overleftrightarrow{1})\cdot\boldsymbol{nn}]+(1-\cos\alpha)\cos\beta\boldsymbol{nn} \\
&+ \cos\alpha\cos\beta\overleftrightarrow{1}+\sin\alpha\cos\beta(\boldsymbol{n}\times\overleftrightarrow{1})+(1-\cos\alpha)\sin\beta[\boldsymbol{nn}\cdot(\boldsymbol{n}\times\overleftrightarrow{1})] \\
&+ \cos\alpha\sin\beta(\boldsymbol{n}\times\overleftrightarrow{1})+\sin\alpha\sin\beta(\boldsymbol{n}\times\overleftrightarrow{1})\cdot(\boldsymbol{n}\times\overleftrightarrow{1}) \\
&= \boldsymbol{nn}[1-\cos\alpha-\cos\beta+\cos\alpha\cos\beta+\cos\alpha-\cos\alpha\cos\beta \\
&+ \cos\beta-\cos\alpha\cos\beta+\sin\alpha\sin\beta]+\overleftrightarrow{1}(\cos\alpha\cos\beta-\sin\alpha\sin\beta) \\
&+ \boldsymbol{n}\times\overleftrightarrow{1}(\sin\alpha\cos\beta+\cos\alpha\sin\beta) \\
&= [1-\cos(\alpha+\beta)]\boldsymbol{nn}+\cos(\alpha+\beta)\overleftrightarrow{1}+\sin(\alpha+\beta)(\boldsymbol{n}\times\overleftrightarrow{1}) \\
&= \overleftrightarrow{R}[\boldsymbol{n},(\alpha+\beta)]
\end{aligned}$$

4)

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta) = \overleftrightarrow{1} \tag{6.73}$$

Here $\overleftrightarrow{R}^T(\boldsymbol{n}, \beta)$ is the transpose of $\overleftrightarrow{R}$ and carries the similar sense as in matrix notation. In the operator $\overleftrightarrow{R}$, $\boldsymbol{nn}$ is symmetric and the transpose of $\boldsymbol{n} \times \overleftrightarrow{1}$ gives $-(\boldsymbol{n} \times \overleftrightarrow{1})$; hence,

$$\overleftrightarrow{R}^T(\boldsymbol{n}, \beta) = (1 - \cos\beta)\boldsymbol{nn} + \cos\beta\overleftrightarrow{1} - \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) = \overleftrightarrow{R}(\boldsymbol{n}, -\beta)$$

and

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta) = \overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \overleftrightarrow{R}(\boldsymbol{n}, -\beta)$$

$$= \overleftrightarrow{R}(\boldsymbol{n}, \beta - \beta) = \overleftrightarrow{R}(\boldsymbol{n}, 0) = \overleftrightarrow{1}$$

5)

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \boldsymbol{V} = \boldsymbol{V} \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta) \tag{6.74}$$

Proof:

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \boldsymbol{V} = [(1 - \cos\beta)\boldsymbol{nn} + \cos\beta\overleftrightarrow{1} + \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1})] \cdot \boldsymbol{V}$$

$$= (1 - \cos\beta)\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{V}) + \cos\beta\boldsymbol{V} + \sin\beta(\boldsymbol{n} \times \boldsymbol{V})$$

$$= (1 - \cos\beta)(\boldsymbol{V} \cdot \boldsymbol{n})\boldsymbol{n} + \cos\beta\boldsymbol{V} \cdot \overleftrightarrow{1} - \sin\beta\boldsymbol{V} \cdot (\boldsymbol{n} \times \overleftrightarrow{1})$$

$$= \boldsymbol{V} \cdot \overleftrightarrow{R}(\boldsymbol{n}, -\beta) = \boldsymbol{V} \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta)$$

6)

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \overleftrightarrow{T} \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta) = \overleftrightarrow{T}' \tag{6.75}$$

Proof: Because

$$\overleftrightarrow{T} = T_{11}\boldsymbol{ii} + T_{12}\boldsymbol{ij} + T_{13}\boldsymbol{ik} + T_{21}\boldsymbol{ji} + \cdots$$

each term in the preceding equation may be represented by $\boldsymbol{AB}$. Without losing generality, let us consider $\overleftrightarrow{T} = \boldsymbol{AB}$, then

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \overleftrightarrow{T} \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta) = \overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \boldsymbol{AB} \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta)$$

$$= (\overleftrightarrow{R} \cdot \boldsymbol{A})(\boldsymbol{B} \cdot \overleftrightarrow{R}^T)$$

$$= \boldsymbol{A}'(\overleftrightarrow{R} \cdot \boldsymbol{B}) \qquad \text{[Eq. (6.74) used]}$$

$$= \boldsymbol{A}'\boldsymbol{B}' = \overleftrightarrow{T}'$$

7)

$$[\overleftrightarrow{R}(\boldsymbol{n},\beta)\cdot \boldsymbol{V}]\times \overleftrightarrow{1} = \overleftrightarrow{R}(\boldsymbol{n},\beta)\cdot(\boldsymbol{V}\times\overleftrightarrow{1})\cdot\overleftrightarrow{R}^T(\boldsymbol{n},\beta) \tag{6.76}$$

Proof:

$$(\overleftrightarrow{R}\cdot \boldsymbol{V})\times\overleftrightarrow{1} = \boldsymbol{V}'\times\overleftrightarrow{1} = (V_1'\boldsymbol{i}' + V_2'\boldsymbol{j}' + V_3'\boldsymbol{k}')\times\overleftrightarrow{1}'$$

in which $\overleftrightarrow{1}' = \overleftrightarrow{R}\cdot\overleftrightarrow{1}\cdot\overleftrightarrow{R}^T = \overleftrightarrow{R}\cdot\overleftrightarrow{R}^T = \overleftrightarrow{1}$ has been used. Hence

$$(\overleftrightarrow{R}\cdot \boldsymbol{V})\times\overleftrightarrow{1} = -V_3'\boldsymbol{i}'\boldsymbol{j}' + V_2'\boldsymbol{i}'\boldsymbol{k}' + V_3'\boldsymbol{j}'\boldsymbol{i}' - V_1'\boldsymbol{j}'\boldsymbol{k}' - V_2'\boldsymbol{k}'\boldsymbol{i}' + V_1'\boldsymbol{k}'\boldsymbol{j}'$$

$$= (\boldsymbol{V}\times\overleftrightarrow{1})' = \overleftrightarrow{R}\cdot(\boldsymbol{V}\times\overleftrightarrow{1})\cdot\overleftrightarrow{R}^T$$

Eq. (6.75) is used in the last step.

8) If a unit vector $\boldsymbol{n}$ is rotated to $\boldsymbol{n}'$ by $\overleftrightarrow{R}(\boldsymbol{m},\alpha)$

$$\boldsymbol{n}' = \overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}$$

then

$$\overleftrightarrow{R}'(\boldsymbol{n}',\beta) = \overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\overleftrightarrow{R}(\boldsymbol{n},\beta)\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha) \tag{6.77}$$

Proof: The relationship between $\overleftrightarrow{R}(\boldsymbol{n}',\beta)$ and $\overleftrightarrow{R}(\boldsymbol{m},\alpha)$ is

$$\overleftrightarrow{R}(\boldsymbol{n}',\beta) = \overleftrightarrow{R}[\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n},\beta]$$

$$= (1-\cos\beta)[\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}][\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}]$$

$$+\ \cos\beta\overleftrightarrow{1} + \sin\beta[\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}]\times\overleftrightarrow{1}$$

Using Eqs. (6.74) and (6.76), we have

$$\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n} = \boldsymbol{n}\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)$$

and

$$[\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}]\times\overleftrightarrow{1} = \overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot(\boldsymbol{n}\times\overleftrightarrow{1})\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)$$

Then

$$\overleftrightarrow{R}(\boldsymbol{n}',\beta) = (1-\cos\beta)[\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}][\boldsymbol{n}\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)]$$

$$+\ \cos\beta\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha) + \sin\beta\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot(\boldsymbol{n}\times\overleftrightarrow{1})\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)$$

$$= \overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot[(1-\cos\beta)\boldsymbol{n}\boldsymbol{n} + \cos\beta\overleftrightarrow{1} + \sin\beta(\boldsymbol{n}\times\overleftrightarrow{1})]\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)$$

$$= \overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\overleftrightarrow{R}(\boldsymbol{n},\beta)\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)$$

Applications of the Rotation Operator

Rotation of coordinate system through Euler angles ϕ, θ, and ψ. Suppose that we rotate the coordinate first with respect to $\boldsymbol{k}$ by an angle of ϕ. The position vector $\boldsymbol{r}$ is rotated with the rotation of coordinates. The new vector $\boldsymbol{r}'$ can be expressed as

$$\boldsymbol{r}' = \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \boldsymbol{r} \tag{6.78}$$

Note that this operation is not the same as in the operation of a rotation matrix

$$\boldsymbol{r}'_m = \boldsymbol{R}_1 \boldsymbol{r} \tag{6.79}$$

where $\boldsymbol{r}'_m$ is the vector $\boldsymbol{r}$ in the rotated coordinates; $\boldsymbol{r}$ itself is not rotated. To emphasize this difference, let us consider

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{i} \\ \overleftrightarrow{R}_1\left(\boldsymbol{k}, \frac{\pi}{2}\right) &= \boldsymbol{kk} + (\boldsymbol{ji} - \boldsymbol{ij}) \\ \boldsymbol{r}' = \boldsymbol{i}' &= (\boldsymbol{kk} + \boldsymbol{ji} - \boldsymbol{ij}) \cdot \boldsymbol{i} = \boldsymbol{j} \end{aligned} \tag{6.80}$$

This means the vector $\boldsymbol{i}$ is rotated to $\boldsymbol{j}$ after the coordinate axis $\boldsymbol{i}$ is rotated about $\boldsymbol{k}$ by an angle of $\pi/2$. On the other hand, the operation of Eq. (6.79) by rotation matrix will have a totally different result. Let us see the following case:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{R}_1 = \begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} & 0 \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{r}'_m = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \tag{6.81}$$

This $x'_2 = -1$ means that the vector $\boldsymbol{i}$ is not moved and is now along $-\boldsymbol{j}'$ after the whole coordinate system is rotated about $\boldsymbol{k}$ by an angle of $\pi/2$. The operation of a rotation matrix also can be expressed as a dyadic operation:

$$\boldsymbol{r}'_m = \overleftrightarrow{R}_1 \cdot \boldsymbol{r} \tag{6.82}$$

in which $\overleftrightarrow{R}_1 = \boldsymbol{i}'\boldsymbol{j} - \boldsymbol{j}'\boldsymbol{i} + \boldsymbol{k}'\boldsymbol{k}$, and $\boldsymbol{r} = \boldsymbol{i}$. Hence, in the rotated coordinate system, the unit vector becomes

$$\boldsymbol{r}'_m = -\boldsymbol{j}' \tag{6.83}$$

Now, continuing to consider the rotation of position vector $\boldsymbol{r}'$ with the coordinates from Eq. (6.78), let us rotate $\boldsymbol{r}'$ about $\boldsymbol{i}'$ by an angle of θ. Then we have

$$\boldsymbol{r}'' = \overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \boldsymbol{r}' = \overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \boldsymbol{r}$$

Next we rotate $\boldsymbol{r}''$ about $\boldsymbol{k}''$ by an angle of ψ, and we find

$$\begin{aligned} \boldsymbol{r}''' &= \overleftrightarrow{R}_3(\boldsymbol{k}'', \psi) \cdot \boldsymbol{r}'' \\ &= \overleftrightarrow{R}_3(\boldsymbol{k}'', \psi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \boldsymbol{r} \end{aligned} \tag{6.84}$$

where $\boldsymbol{r}'''$ is the final form of the position vector $\boldsymbol{r}$ after being rotated about $\boldsymbol{k}$ by angle of ϕ, rotated about $\boldsymbol{i}'$ by θ and rotated about $\boldsymbol{k}''$ by ψ . Note that $\boldsymbol{i}'$ and $\boldsymbol{k}''$ are unit vectors along rotated coordinates. It will be more convenient to rotate $\boldsymbol{r}$ with respect to fixed axes. With the use of Eq. (6.77), we can express

$$\overleftrightarrow{R}_2(\boldsymbol{i}', \theta) = \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \cdot \overleftrightarrow{R}_1^T(\boldsymbol{k}, \phi)$$

Taking the dot product with $\overleftrightarrow{R}_1(\boldsymbol{k}, \phi)$ from the right leads to

$$\overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) = \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \tag{6.85}$$

Similarly,

$$\begin{aligned} \overleftrightarrow{R}_3(\boldsymbol{k}'', \psi) &= [\overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi)] \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi) \cdot [\overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi)]^T \\ &= [\overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi)] \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi) \cdot \left[\overleftrightarrow{R}_1^T(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2^T(\boldsymbol{i}', \theta)\right] \end{aligned}$$

Multiplying from the right by $\overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_1$ gives

$$\begin{aligned} \overleftrightarrow{R}_3(\boldsymbol{k}'', \psi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) &= \overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi) \\ &= \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi) \end{aligned} \tag{6.86}$$

Equation (6.85) has been used in the last step of the manipulation. The result reached in Eq. (6.86) shows that the Euler angles ϕ, θ, ψ can be replaced by rotating the position vector $\boldsymbol{r}$ with respect to unprimed axes in a reversed order of ψ, θ, ϕ.

Applying the preceding results to a vector $\boldsymbol{r}$ fixed in space but with the coordinate system rotated, the relation between primed system and unprimed system may be derived by

$$\boldsymbol{r} = x_1'''\boldsymbol{i}''' + x_2'''\boldsymbol{j}''' + x_3'''\boldsymbol{k}''' = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$$

where

$$\boldsymbol{i}''' = \overleftrightarrow{R} \cdot \boldsymbol{i}, \qquad \boldsymbol{j}''' = \overleftrightarrow{R} \cdot \boldsymbol{j}, \qquad \boldsymbol{k}''' = \overleftrightarrow{R} \cdot \boldsymbol{k} \tag{6.87}$$

or

$$r = \sum_k x_k''' \varepsilon_k''' = \sum_j x_j \varepsilon_j$$

Taking the dot product of the preceding equation with ε_i''' leads to

$$x_i''' = \sum_k x_k''' (\varepsilon_k''' \cdot \varepsilon_i''') = \sum_j x_j (\varepsilon_j \cdot \varepsilon_i''') = \sum_j x_j (\varepsilon_j \cdot \overleftrightarrow{R} \cdot \varepsilon_i)$$

Note that $\varepsilon_j \cdot \overleftrightarrow{R} \cdot \varepsilon_i = \varepsilon_j \cdot \varepsilon_i'''$ = direction cosine between ε_j and ε_i'''. Therefore

$$a_{ij} = \varepsilon_j \cdot \overleftrightarrow{R} \cdot \varepsilon_i$$

For $\overleftrightarrow{R} = \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi)$,

$$\begin{aligned} a_{ij} &= \varepsilon_j \cdot \overleftrightarrow{R}_1 \cdot \overleftrightarrow{1} \cdot \overleftrightarrow{R}_2 \cdot \overleftrightarrow{1} \cdot \overleftrightarrow{R}_3 \cdot \varepsilon_i \\ &= \sum_{k,\ell} [\varepsilon_j \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \varepsilon_k][\varepsilon_k \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \cdot \varepsilon_\ell][\varepsilon_\ell \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi) \cdot \varepsilon_i] \end{aligned} \tag{6.88}$$

The preceding equation can be easily used to verify that the result agrees well with $(a_{ij}) = \boldsymbol{R}_3\boldsymbol{R}_2\boldsymbol{R}_1$ given in Eq. (6.16).

Combination of two successive rotations about different axes by one rotation. Suppose a rigid body to be rotated by two steps. First it is rotated about the $\boldsymbol{k}'$ axis by an angle of ϕ and then it is rotated about the $\boldsymbol{k}$ axis by an angle of ψ. The directions of $\boldsymbol{k}$ and $\boldsymbol{k}'$ are known, and the plane containing them is

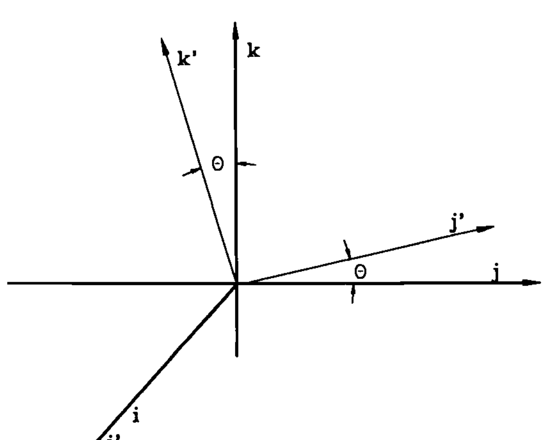

Fig. 6.10 True angle θ between axes $\boldsymbol{k}$ and $\boldsymbol{k}'$.

determined. Choose the x axis perpendicular to the plane. Suppose the true angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$ is θ, as shown in Fig. 6.10, then

$$\boldsymbol{k}' = \overleftrightarrow{R} \cdot \boldsymbol{k} = -(\sin\theta)\boldsymbol{j} + (\cos\theta)\boldsymbol{k} \tag{6.89}$$

And the two consecutive rotations may be expressed by

$$\overleftrightarrow{R}_1 = (1 - \cos\phi)\boldsymbol{k}\boldsymbol{k} + \cos\phi\overleftrightarrow{1} + \sin\phi(\boldsymbol{k} \times \overleftrightarrow{1})$$

and

$$\overleftrightarrow{R}_2 = (1 - \cos\psi)\boldsymbol{k}'\boldsymbol{k}' + \cos\psi\overleftrightarrow{1} + \sin\psi(\boldsymbol{k}' \times \overleftrightarrow{1})$$

According to Euler's theorem that the most general displacement of a rigid body with one point fixed is equivalent to a single rotation about some axis through that point, these two rotations can be combined into one, i.e.,

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) = \overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_1 \tag{6.90}$$

The theorem is established if $\boldsymbol{n}$ and β are determined uniquely. To determine them, let us start from

$$(1 - \cos\beta)\boldsymbol{n}\boldsymbol{n} + \cos\beta\overleftrightarrow{1} + \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) = \overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_1$$

and taking the transpose of both sides,

$$(1 - \cos\beta)\boldsymbol{n}\boldsymbol{n} + \cos\beta\overleftrightarrow{1} - \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) = (\overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_1)^T = \overleftrightarrow{R}_1^T \cdot \overleftrightarrow{R}_2^T$$

The subtraction of the preceding two equations gives

$$\sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) = \tfrac{1}{2}\left[\overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_1 - \overleftrightarrow{R}_1^T \cdot \overleftrightarrow{R}_2^T\right] \tag{6.91}$$

After the right hand of the equation is expanded in detail, the following identities are used for simplification:

$$\boldsymbol{i} \times \overleftrightarrow{1} = \boldsymbol{k}\boldsymbol{j} - \boldsymbol{j}\boldsymbol{k}, \qquad \boldsymbol{j} \times \overleftrightarrow{1} = \boldsymbol{i}\boldsymbol{k} - \boldsymbol{k}\boldsymbol{i}, \qquad \boldsymbol{k} \times \overleftrightarrow{1} = \boldsymbol{j}\boldsymbol{i} - \boldsymbol{i}\boldsymbol{j},$$

$$(\boldsymbol{k}' \times \overleftrightarrow{1}) \cdot \boldsymbol{k} = \boldsymbol{k}' \times \boldsymbol{k} = -\sin\theta\boldsymbol{i}$$

$$(\boldsymbol{k} \times \overleftrightarrow{1}) \cdot (\boldsymbol{k}' \times \overleftrightarrow{1}) = (\boldsymbol{k} \times \overleftrightarrow{1}) \times \boldsymbol{k}' = -\boldsymbol{j}\boldsymbol{j}' - \boldsymbol{i}\boldsymbol{i}'\cos\theta$$

and

$$\boldsymbol{j}' = \overleftrightarrow{R}(\boldsymbol{i}, \theta) \cdot \boldsymbol{j} = \cos\theta\boldsymbol{j} + \sin\theta\boldsymbol{k}$$

Finally Eq. (6.91) is reduced to

$$\sin\beta(\boldsymbol{n}\times\overleftrightarrow{1}) = 2\left[\cos\frac{\psi}{2}\cos\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\phi}{2}\cos\theta\right]\cdot\left[\cos\frac{\psi}{2}\sin\frac{\phi}{2}(\boldsymbol{k}\times\overleftrightarrow{1})\right.$$

$$\left. + \sin\frac{\psi}{2}\cos\frac{\phi}{2}(\boldsymbol{k}'\times\overleftrightarrow{1}) + \sin\frac{\psi}{2}\sin\frac{\phi}{2}(\boldsymbol{k}'\times\boldsymbol{k})\times\overleftrightarrow{1}\right]$$

$$= 2\sin\frac{\beta}{2}\cos\frac{\beta}{2}(\boldsymbol{n}\times\overleftrightarrow{1}) \tag{6.92}$$

To identify the β and $\boldsymbol{n}$ in the preceding equation, let us consider a special case of $\theta = 0$, then $\boldsymbol{k} = \boldsymbol{k}' = \boldsymbol{n}$:

$$\sin\frac{\beta}{2}\cos\frac{\beta}{2}(\boldsymbol{k}\times\overleftrightarrow{1}) = \cos\left(\frac{\psi+\phi}{2}\right)\cdot\sin\left(\frac{\psi+\phi}{2}\right)(\boldsymbol{k}\times\overleftrightarrow{1})$$

Hence

$$\cos\frac{\beta}{2} = \cos\frac{\psi}{2}\cos\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\phi}{2}\cos\theta \tag{6.93}$$

$$\boldsymbol{n} = \frac{1}{\sin(\beta/2)}\left[\cos\frac{\psi}{2}\sin\frac{\phi}{2}\boldsymbol{k} + \sin\frac{\psi}{2}\cos\frac{\phi}{2}\boldsymbol{k}' + \sin\frac{\psi}{2}\sin\frac{\phi}{2}(\boldsymbol{k}'\times\boldsymbol{k})\right] \tag{6.94}$$

Because $\boldsymbol{n}$ and β are properly determined, Euler's theorem, that two consecutive rotations with respect to two different axes can be combined into one rotational movement, is proved. Let us use an example to illustrate this concept as follows.

Example 6.4

A slab of size $a \times b$ and a thickness of t is placed vertically in x–z plane at the beginning. Suppose that the slab experiences two different kinds of rotations while the point at the origin remains fixed. Consider two different cases: 1) the slab is rotated first about the x axis by 90 deg, then about the y' axis by 90 deg; 2) the slab is rotated first about the y axis by 90 deg and then about the x' axis by 90 deg.

1) Perform the rotations through two steps.

2) Combine the rotations into one step and show the same results reached as in part 1.

Solution. 1a) The slab is rotated about the x axis by 90 deg then about the y' axis by 90 deg. At the beginning the unit normal vector of the slab is denoted by $\boldsymbol{n}'$ which is parallel to the y axis, or $\boldsymbol{n} = \boldsymbol{j}$ (see Fig. 6.11).

After it is rotated about the x axis by 90 deg, the normal vector becomes

$$\boldsymbol{n}' = \overleftrightarrow{R}(\boldsymbol{i}, 90\,\text{deg})\cdot\boldsymbol{j}$$

$$= (\boldsymbol{i}\boldsymbol{i} + \boldsymbol{i}\times\overleftrightarrow{1})\cdot\boldsymbol{j} = \boldsymbol{k}, \qquad \boldsymbol{n}' = \boldsymbol{j}' = \boldsymbol{k}$$

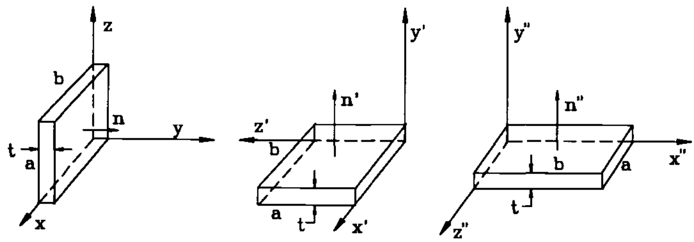

Fig. 6.11 Slab is rotated according to case 1a.

Finally, after the second rotation about the y' axis by 90 deg, the new normal vector is

$$\boldsymbol{n}'' = \overleftrightarrow{R}(\boldsymbol{j}', 90\text{ deg}) \cdot \boldsymbol{k} = \overleftrightarrow{R}(\boldsymbol{k}, 90\text{ deg}) \cdot \boldsymbol{k} = (\boldsymbol{k}\boldsymbol{k} + \boldsymbol{k} \times \overleftrightarrow{1}) \cdot \boldsymbol{k} = \boldsymbol{k} \tag{6.95}$$

1b) The slab is rotated about the y axis by 90 deg then about the x' axis by 90 deg (see Fig. 6.12). Similarly, as in case 1a, the normal vectors of the slab are denoted by $\boldsymbol{n}$, $\boldsymbol{n}'$, and $\boldsymbol{n}''$ for three positions of the slab. We find

$$\boldsymbol{n} = \boldsymbol{j}$$

$$\boldsymbol{n}' = \overleftrightarrow{R}(\boldsymbol{j}, 90\text{ deg}) \cdot \boldsymbol{j} = \boldsymbol{j}, \qquad \boldsymbol{i}' = -\boldsymbol{k}$$

and

$$\boldsymbol{n}'' = \overleftrightarrow{R}(\boldsymbol{i}', 90\text{ deg}) \cdot \boldsymbol{j} = \overleftrightarrow{R}(-\boldsymbol{k}, 90\text{ deg}) \cdot \boldsymbol{j} = (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{k} \times \overleftrightarrow{1}) \cdot \boldsymbol{j} = \boldsymbol{i} \tag{6.96}$$

2a) The slab is rotated with respect to $\boldsymbol{n}_1$ and by an angle of β_1 for the case of 1a:

$$\cos\frac{\beta_1}{2} = \cos^2 45\text{ deg} - \sin^2 45\text{ deg}\cos 90\text{ deg} = \frac{1}{2}$$

$$\frac{\beta_1}{2} = 60\text{ deg} \qquad \beta_1 = 120\text{ deg}$$

$$\boldsymbol{n}_1 = \frac{1}{\sin(\beta/2)}[\cos 45\text{ deg}\sin 45\text{ deg}\,\boldsymbol{i} + \sin 45\text{ deg}\cos 45\text{ deg}\,\boldsymbol{j}'$$
$$+ \sin^2 45\text{ deg}(\boldsymbol{j}' \times \boldsymbol{i})] = \frac{1}{\sqrt{3}/2}\left[\frac{1}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{k} + \frac{1}{2}\boldsymbol{j}\right] = \frac{1}{\sqrt{3}}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$

$$\boldsymbol{n}'' = \overleftrightarrow{R}(\boldsymbol{n}_1, 120\text{ deg}) \cdot \boldsymbol{j} = [(1 - \cos 120\text{ deg})\boldsymbol{n}_1\boldsymbol{n}_1$$
$$+ \cos 120\text{ deg}\,\overleftrightarrow{1} + \sin 120\text{ deg}(\boldsymbol{n}_1 \times \overleftrightarrow{1})] \cdot \boldsymbol{j}$$
$$= \frac{3}{2}\boldsymbol{n}_1\frac{1}{\sqrt{3}} - \frac{1}{2}\boldsymbol{j} + \frac{1}{2}(\boldsymbol{k} - \boldsymbol{i}) = \boldsymbol{k}$$

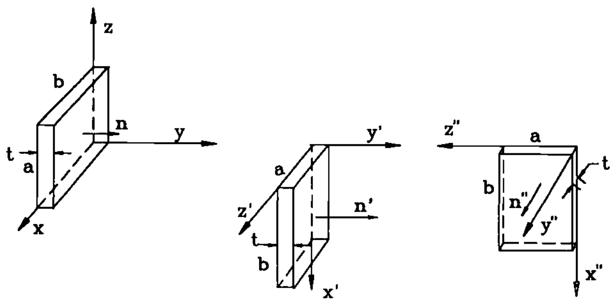

Fig. 6.12 Slab is rotated according to case 1b.

The same result is reached as given in Eq. (6.95), but here is done only by one rotation.

2b) The slab is rotated with respect to $\boldsymbol{n}_2$ by angle of β_2 for case 1b:

$$\cos\frac{\beta_2}{2} = \cos^2 45\,\text{deg} - \sin^2 45\,\text{deg}\cos 90\,\text{deg} = \frac{1}{2}, \qquad \beta_2 = 120\,\text{deg}$$

$$\boldsymbol{n}_2 = \frac{1}{\sin(\beta/2)}[\cos 45\,\text{deg}\sin 45\,\text{deg}\,\boldsymbol{j} + \sin 45\,\text{deg}\cos 45\,\text{deg}\,\boldsymbol{i}' + \sin^2 45\,\text{deg}(\boldsymbol{i}' \times \boldsymbol{j})] = \frac{1}{\sqrt{3}}(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$$

$$\boldsymbol{n}'' = \overleftrightarrow{R}(\boldsymbol{n}_2, 120\,\text{deg}) \cdot \boldsymbol{j} = [(1 - \cos 120\,\text{deg})\boldsymbol{n}_2\boldsymbol{n}_2 + \cos 120\,\text{deg}\overleftrightarrow{1} + \sin 120\,\text{deg}(\boldsymbol{n}_2 \times \overleftrightarrow{1}] \cdot \boldsymbol{j} = \frac{3}{2}\boldsymbol{n}_2\frac{1}{\sqrt{3}} - \frac{1}{2}\boldsymbol{j} + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}}(\boldsymbol{k} + \boldsymbol{i}) = \boldsymbol{i}$$

The same result is found as given in Eq. (6.96) for two steps of rotation. More details may be shown if the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are rotated as $\boldsymbol{n}$. That approach has been assigned as an exercise for readers to complete in the problems section.

Problems

6.1. Verify that the following transformations are orthogonal:

(a)

$$x' = (\cos\theta)x + (\sin\theta)y$$
$$y' = (-\sin\theta)x + (\cos\theta)y$$

(b)

$$x_1' = \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_3$$

$$x_2' = x_2$$

$$x_3' = -\frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_3$$

6.2. Prove that the product of two orthogonal transformations is an orthogonal transformation.

6.3. Given a stress matrix

$$\begin{pmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{pmatrix} \qquad \text{(ksi)}$$

find the principal stresses and the corresponding principal axes.

6.4. It is known that the moments and products of inertia of area A for the centroidal axes are

$$I_{xx} = 40 \text{ ft}^4, \qquad I_{yy} = 20 \text{ ft}^4, \qquad I_{xy} = -4 \text{ ft}^4$$

Find the principal moments of inertia and the corresponding principal axes in the x–y plane.

6.5. Similar to the derivation of viscous stress in Newtonian fluid as given in Section 6.6, derive the expression of stress tensor in homogeneous solid as a function of strains.

6.6. Prove that

$$\boldsymbol{A}\cdot(\boldsymbol{n} \times \overleftrightarrow{1}) = \boldsymbol{A} \times \boldsymbol{n}$$

and

$$(\boldsymbol{n} \times \overleftrightarrow{1})\cdot\boldsymbol{A} = \boldsymbol{n} \times \boldsymbol{A}$$

6.7. Prove that

$$(\boldsymbol{n} \times \overleftrightarrow{1})\cdot(\boldsymbol{n} \times \overleftrightarrow{1}) = \boldsymbol{n}\boldsymbol{n} - \overleftrightarrow{1}$$

6.8. Suppose that the angle between two unit vectors $\boldsymbol{k}$ and $\boldsymbol{k}'$ is θ as shown in Fig. 6.10. Prove that

$$(\boldsymbol{k} \times \overleftrightarrow{1}) \cdot (\boldsymbol{k}' \times \overleftrightarrow{1}) = (\boldsymbol{k} \times \overleftrightarrow{1}) \times \boldsymbol{k}' = -\boldsymbol{j}\boldsymbol{j}' - \boldsymbol{i}\boldsymbol{i}' \cos\theta$$

6.9. A slab of size $a \times b$ and thickness of t is placed vertically in x–z plane at the beginning as shown in Fig. 6.11. Suppose that the slab is rotated but with the point at the origin fixed. First, the slab is rotated about the x axis by 30 deg then rotated about the y' axis by 60 deg.

(a) Perform the rotations through two steps.

(b) Combine the rotations into one step and show the same results reached as in part (a).

6.10. Consider Example 6.4. Let $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ be the unit vectors of the initial coordinate system. The vectors are $\boldsymbol{i}'\boldsymbol{j}'$ and $\boldsymbol{k}'$ after the first rotation and the unit vectors are $\boldsymbol{i}''\boldsymbol{j}''$ and $\boldsymbol{k}''$ after the second rotation. Find the relationships between these unit vectors for the rotations considered in the example.

6.11. Suppose that $\boldsymbol{I}_{xx}, \boldsymbol{I}_{yy}$, and $\boldsymbol{I}_{zz}$ are given and the products of inertia are zero. Find the moment of inertia matrix when the coordinate system is rotated about the z axis by an angle of θ.

7
Dynamics of a Rigid Body

A RIGID body is a body with finite volume, mass, and shape that remains unchanged during the observation. Deformation of the body is not considered in this chapter. When a force and torque are applied to a rigid body, translational and rotational motions of the body will take place and are studied in this chapter. Because most objects can be modeled as a rigid body, the analysis of rigid-body dynamics is very useful and is the major subject of this book. Many general principles for the dynamics of particles studied in the preceding chapters provide necessary background for this chapter. Matrices and rotational operators from Chapter 6 are used extensively and should be reviewed before studying this chapter.

Fundamental principles are given in the first three sections, followed by three sections of specific examples. Section 7.1 introduces the general concept of a solid body in motion and explains how any motion always can be treated as a combination of translational and rotational motions. Section 7.2 derives the equation of motion for a mass in a moving frame of reference, which is, in general, motion relative to an inertial frame of reference. The foundation of the relations is known as Galilean transformation. Section 7.3 describes how to obtain the Euler's angular velocity using two different approaches: one uses matrix operation and the other the rotation operator. Both of them reach the same result. The difference between them is that, while the rotation operator rotates the position vector (as in Chapter 6), matrix operation uses the rotation of coordinates, not the vector. The use of these two approaches demonstrates how divergent methods can achieve the same result and also shows the usefulness of the rotation operator. Because the rotation operator was only recently rediscovered, its many applications have yet to be developed. A simple example for Euler's equations of motion is included in this section.

The second half of this chapter uses the physical concepts presented in the preceding chapters to solve both classical and contemporary problems. In Section 7.4 we deal with gyroscopic motion and use three examples for studying its fundamental principles. The first example demonstrates that a rotating propeller (or other rotating mechanisms such as turbines and compressors) can produce gyroscopic force, which tends to cause an airplane to dive or climb during yawing. The second example studies a single-degree-of-freedom gyro. The last example in this section explains the oscillation of the spinning axis in a gyro-compass caused by the Earth's rotation. The oscillation frequency of the axis about the meridian is determined. Section 7.5 is devoted to studying the motion of a heavy symmetrical top. The nutation and precession of the spinning axis are analyzed in detail. The nutation angle vs precession angle for three possible cases is integrated, and the results are presented. Section 7.6 studies a satellite in a circular orbit using the equation derived in Section 7.2 for a solid body in motion. This is the first example involving a solid body in general motion. The results of this study show that the yawing and rolling motions of a satellite always will generate torques about all three axes. We have only recently entered the space era and still must solve many dynamics problems related to the motion of space vehicles. This section opens that door.

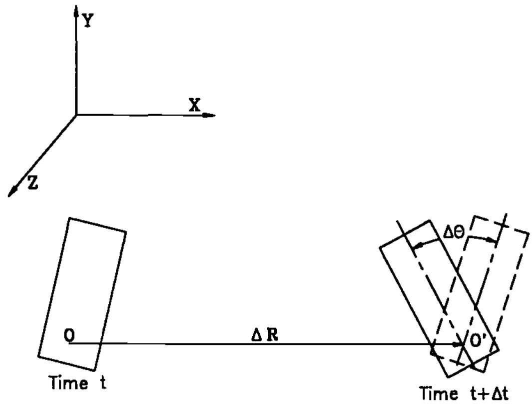

Fig. 7.1 General motion of a rigid body.

The examples included here are simple in comparison with many problems facing engineers today. However, I hope that the discussion and examples of this book will stimulate interest in and provide a firm foundation for further study and research work in this area.

7.1 Displacements of a Rigid Body

In three-dimensional space, six degrees of freedom are needed to specify the position of a solid body. Consider a coordinate system in translational motion with the body. Three degrees of freedom describe the origin of the coordinates and three degrees of freedom describe the rotational displacements of the body with respect to the three axes. As seen in earlier chapters, the origin of the moving coordinates is usually fixed at the center of mass of the body in order to simplify the equations. In certain cases, however, it is more convenient to place the origin of the moving coordinates elsewhere.

All rigid body motion can be reduced to translation combined with rotational motion as shown in Fig. 7.1. This is known as Chasles's theorem. If one point of the body is fixed, then the motion must be rotational only. The rotational displacements, no matter how complicated, always can be expressed by one rotation of the body with respect to an axis through the fixed point. This is known as Euler's theorem. In Section 6.7, we proved that two successive rotations about the axes through zero can be combined to a single rotation about an axis through zero. By a repeated application of that result, any number of successive rotations about the same point can be reduced to one rotation. This is another statement of Euler's theorem.

7.2 Relationship Between Derivatives of a Vector for Different Reference Frames

Vector in Moving Reference Frame Rotating Relative to Fixed or Inertial Reference System

Consider that xyz is a moving reference that rotates relative to a fixed reference denoted by XYZ. A vector $\boldsymbol{G}$ in the moving system can be expressed as

$$\boldsymbol{G} = \sum_i G_i \boldsymbol{e}_i$$

where e_i is a unit vector in the rotating system. G also can be expressed in the fixed system. Thus the time derivative of G, as seen from the fixed system, is obtained as

$$\left(\frac{dG}{dt}\right)_{\text{fixed}} = \sum_i \dot{G}_i e_i + \sum_i G_i \dot{e}_i$$

However,

$$\sum_i \dot{G}_i e_i = \left(\frac{dG}{dt}\right)_{\text{rotating}} = \left(\frac{dG}{dt}\right)_{xyz}$$

where $(dG/dt)_{xyz}$ means the rate change of G as observed in the rotating system. On the other hand, in the fixed system the velocity of a point fixed in the rotating system is

$$v = \omega \times r \qquad \text{as } r = e_i$$

$$\dot{e}_i = \omega \times e_i$$

Hence

$$\sum_i G_i \dot{e}_i = \omega \times G$$

$$\left(\frac{dG}{dt}\right)_{\text{fixed}} = \left(\frac{dG}{dt}\right)_{XYZ} = \left(\frac{dG}{dt}\right)_{\text{rotating}} + \omega \times G = \left(\frac{dG}{dt}\right)_{xyz} + \omega \times G \qquad (7.1)$$

Any vector G differentiated in the fixed coordinates equals the change of G in the rotating system plus $\omega \times G$ in the rotating system.

Velocities and Accelerations of a Particle in Different References

Suppose that XYZ is a fixed or inertial reference; xyz is a moving reference that is in both translational and rotational motion, as shown in Fig. 7.2. R is the position vector of the origin of xyz system and r and r' are position vectors of point P in XYZ and xyz systems, respectively.

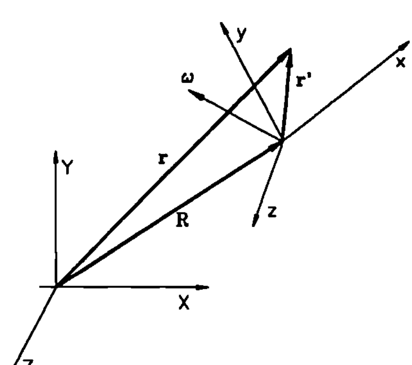

Fig. 7.2 Moving reference system relative to the inertial frame of reference.

Hence

$$r = R + r'$$

Differentiating with respect to time for the XYZ reference, we have

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)_{XYZ} = V_{XYZ} = \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)_{XYZ} + \left(\frac{\mathrm{d}r'}{\mathrm{d}t}\right)_{XYZ}$$

$$= \dot{R} + \left(\frac{\mathrm{d}r'}{\mathrm{d}t}\right)_{xyz} + \omega \times r' = \dot{R} + V_{xyz} + \omega \times r' \tag{7.2}$$

in which Eq. (7.1) has been used in the last step of manipulations. This means that the velocity of point P observed in the fixed reference system equals the vector sum of the velocity of the origin of the moving system, the velocity of point P in the moving system and the velocity of P due to rotation of the xyz system.

Differentiating Eq. (7.2) with respect to time for the XYZ reference system, we get

$$\begin{aligned}\left(\frac{\mathrm{d}V_{XYZ}}{\mathrm{d}t}\right)_{XYZ} &= \ddot{R} + \left(\frac{\mathrm{d}V_{xyz}}{\mathrm{d}t}\right)_{XYZ} + \left[\frac{\mathrm{d}}{\mathrm{d}t}(\omega \times r')\right]_{XYZ} \\ a_{XYZ} &= \ddot{R} + \left(\frac{\mathrm{d}V_{xyz}}{\mathrm{d}t}\right)_{XYZ} + \omega \times \left(\frac{\mathrm{d}r'}{\mathrm{d}t}\right)_{XYZ} + \left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)_{XYZ} \times r'\end{aligned} \tag{7.3}$$

Because

$$\left(\frac{\mathrm{d}V_{xyz}}{\mathrm{d}t}\right)_{XYZ} = \left(\frac{\mathrm{d}V_{xyz}}{\mathrm{d}t}\right)_{xyz} + \omega \times V_{xyz}$$

$$\left(\frac{\mathrm{d}r'}{\mathrm{d}t}\right)_{XYZ} = \left(\frac{\mathrm{d}r'}{\mathrm{d}t}\right)_{xyz} + \omega \times r'$$

Substituting into Eq. (7.3), we find

$$a_{XYZ} = \left(\frac{\mathrm{d}V_{xyz}}{\mathrm{d}t}\right)_{xyz} + \ddot{R} + \omega \times V_{xyz} + \omega \times \left(\frac{\mathrm{d}r'}{\mathrm{d}t}\right)_{xyz}$$
$$+ \omega \times (\omega \times r') + \left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)_{XYZ} \times r'$$

Note that

$$\left(\frac{\mathrm{d}V_{xyz}}{\mathrm{d}t}\right)_{xyz} = a_{xyz}, \qquad \left(\frac{\mathrm{d}r'}{\mathrm{d}t}\right)_{xyz} = V_{xyz}, \qquad \left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)_{XYZ} = \dot{\omega}$$

Hence, we obtain

$$a_{XYZ} = a_{xyz} + \ddot{R} + 2\omega \times V_{xyz} + \omega \times (\omega \times r') + \dot{\omega} \times r' \tag{7.4}$$

where ω and $\dot{\omega}$ are the angular velocity and acceleration, respectively, of the xyz reference relative to the XYZ reference.

By Newton's law, the force on a particle is

$$\boldsymbol{F} = m\boldsymbol{a}_{XYZ}$$

In the moving reference, we find

$$m\boldsymbol{a}_{xyz} = \boldsymbol{F} - m\ddot{\boldsymbol{R}} - m\dot{\omega} \times \boldsymbol{r}' - m\omega \times (\omega \times \boldsymbol{r}') - 2m\omega \times \boldsymbol{V}_{xyz} \qquad (7.5)$$

This expression gives the effective force acting on the mass as observed in the moving frame of reference. The meaning of each term is explained as follows:

1) The term $-m\ddot{\boldsymbol{R}}$ is the inertial force caused by translational acceleration of the moving frame. For example, during the sudden acceleration of a car, passengers sense the force in the opposite direction to the direction of the acceleration.

2) The term $-m\dot{\omega} \times \boldsymbol{r}'$ is the inertia force produced by angular acceleration of the rotating frame. A mass placed on a rotating disk will experience this inertia force in the direction opposite to the tangential acceleration.

3) The term $-m\omega \times (\omega \times \boldsymbol{r}')$ is the centrifugal force term. When a satellite moves in a circular orbit, this force points outward from the center of Earth and is balanced completely by the gravitational force, causing astronauts to experience weightlessness in the orbit.

4) The term $-2m\omega \times \boldsymbol{V}_{xyz}$ is the Coriolis force, which is the major cause of the counterclockwise rotation of hurricanes in the northern hemisphere. The Earth's rotation causes a component of rotational velocity to point outward from the surface of the Earth. To simplify the problem, let us consider only this component of the rotational velocity. The $-\omega \times \boldsymbol{V}_{xyz}$ will cause the air to move in counterclockwise direction if air moves toward a low pressure center as observed from the top of the low pressure center. The rotational momentum of the air is nearly conserved. Hence, the tangential velocity increases as the air moves closer to the eye of the hurricane.

Apply Eqs. (7.2) and (7.4) to the velocities and accelerations of two points a and b of a rigid body. We imagine the xyz reference embedded in the rigid body with the origin at a as shown in Fig. 7.3. Clearly any point b of the body will

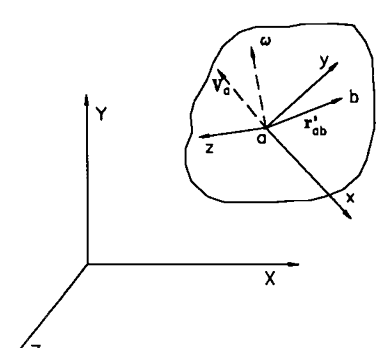

Fig. 7.3 Relative motion between two points in a rigid body in rotation.

not move relative to a and must have $V_{xyz} = 0$ and $a_{xyz} = 0$. Because the origin of xyz corresponds to point $a, \dot{R} = V_a, \ddot{R} = a_a$, velocity and acceleration for point b are

$$V_b = V_a + \omega \times r'_{ab} \tag{7.6}$$

$$a_b = a_a + \omega \times (\omega \times r'_{ab}) + \dot{\omega} \times r'_{ab} \tag{7.7}$$

The preceding two equations are often used in the dynamics of machinery.

7.3 Euler's Angular Velocity and Equations of Motion

Euler's angles have been mentioned in Sections 6.2 and 6.7. They are convenient for describing the motion of a rotating top. Before using them, however, we first need to find the angular velocities for the corresponding angles in three orthogonal coordinates. Many different ways are available to express angular velocity. The most elementary approach is to find the components of $\dot{\phi}, \dot{\theta}$, and $\dot{\psi}$ in the primed system directly; however, it is easy to make a mistake in this approach because $\dot{\phi}, \dot{\theta}$, and $\dot{\psi}$ are not perpendicular. To express these in terms of perpendicular coordinates, the components of $\dot{\phi}, \dot{\theta}$, and $\dot{\psi}$ must be found in the directions of those coordinates, which is not an easy task. The two methods described next are more systematic. By using the rotation matrix, the angular velocities are obtained through simple matrix operations, or the same result can be reached by using the rotation operator that rotates the vector itself (in this case, the unit vectors). These additional applications of the rotation matrix and rotation operator are described in greater detail below.

Euler's Angular Velocity Obtained Through Matrix Operation

Consider a position vector r' that is fixed in the rotating body of $x'''y'''z'''$ and is constant; the corresponding r in the fixed frame of reference is

$$r = R^{-1}r' \tag{7.8}$$

The time derivative of the equation is

$$\dot{r} = \dot{R}^{-1}r' = \dot{R}^T r' = \dot{R}^T R r$$

Applying Eq. (7.1) here for $\dot{r}$ leads to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)_{XYZ} = \omega \times r = [\text{Matrix of}(\omega \times \overleftrightarrow{1})]r$$

Equating the preceding two equations gives a matrix of

$$(\omega \times \overleftrightarrow{1}) = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} = \dot{R}^T R \tag{7.9}$$

To find ω in terms of $\dot{\phi}, \dot{\theta}$, and $\dot{\psi}$, we proceed as follows:

$$\begin{aligned}
\dot{\boldsymbol{R}}^T\boldsymbol{R} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{R}_3\boldsymbol{R}_2\boldsymbol{R}_1)^T\right)(\boldsymbol{R}_3\boldsymbol{R}_2\boldsymbol{R}_1)\\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{R}_1^T\boldsymbol{R}_2^T\boldsymbol{R}_3^T)\right)(\boldsymbol{R}_3\boldsymbol{R}_2\boldsymbol{R}_1)\\
&= \dot{\boldsymbol{R}}_1^T\boldsymbol{R}_1 + \boldsymbol{R}_1^T\dot{\boldsymbol{R}}_2^T\boldsymbol{R}_2\boldsymbol{R}_1 + \boldsymbol{R}_1^T\boldsymbol{R}_2^T\dot{\boldsymbol{R}}_3^T\boldsymbol{R}_3\boldsymbol{R}_2\boldsymbol{R}_1
\end{aligned}$$

These matrix products can be worked out rather simply, for example,

$$\begin{aligned}
\dot{\boldsymbol{R}}_1^T\boldsymbol{R}_1 &= \dot{\phi}\begin{pmatrix} -\sin\phi & -\cos\phi & 0\\ \cos\phi & -\sin\phi & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\\
&= \dot{\phi}\begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\dot{\boldsymbol{R}}_2^T\boldsymbol{R}_2 &= \dot{\theta}\begin{pmatrix} 0 & 0 & 0\\ 0 & -\sin\theta & -\cos\theta\\ 0 & \cos\theta & -\sin\theta\end{pmatrix}\begin{pmatrix} 1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}\\
&= \dot{\theta}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}
\end{aligned}$$

The final result of the matrix algebra is

$$\dot{\boldsymbol{R}}^T\boldsymbol{R}$$

$$= \begin{pmatrix} 0 & -(\dot{\phi}+\dot{\psi}\cos\theta) & (\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi)\\ (\dot{\phi}+\dot{\psi}\cos\theta) & 0 & -(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)\\ -(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi) & (\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi) & 0\end{pmatrix}$$

Therefore, we find from Eq. (7.9)

$$\omega = \begin{pmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi\\ \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi\\ \dot{\phi} + \dot{\psi}\cos\theta\end{pmatrix} \tag{7.10}$$

for the components of angular velocity in the XYZ frame of reference. The velocity component can be expressed in any other primed frame of reference according to the transformation

$$\omega' = \boldsymbol{R}\omega$$

where $\boldsymbol{R}$ can be $\boldsymbol{R}_1$ and $\boldsymbol{R}_2\boldsymbol{R}_1$ or $\boldsymbol{R}_3\boldsymbol{R}_2\boldsymbol{R}_1$.

Euler's Angular Velocity Obtained Through Rotation Operator

Consider a position vector $\boldsymbol{r}$ with initial position $\boldsymbol{r}(0)$. After some time t, it is rotated to $\boldsymbol{r}(t)$, so that

$$\boldsymbol{r}(t) = \overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \boldsymbol{r}(0) \tag{7.11}$$

where

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) = (1 - \cos\beta)\boldsymbol{nn} + \cos\beta\overleftrightarrow{1} + \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1})$$

is the dyadic rotation operator defined in Section 6.7. Taking the time derivative of Eq. (7.11) gives

$$\frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}\overleftrightarrow{R}}{\mathrm{d}t} \cdot \boldsymbol{r}(0) = \frac{\mathrm{d}\overleftrightarrow{R}}{\mathrm{d}t} \cdot \overleftrightarrow{R}^T \cdot \boldsymbol{r}(t) = \omega \times \boldsymbol{r}(t) = (\omega \times \overleftrightarrow{1}) \cdot \boldsymbol{r}(t)$$

which means

$$\omega \times \overleftrightarrow{1} = \frac{\mathrm{d}\overleftrightarrow{R}}{\mathrm{d}t} \cdot \overleftrightarrow{R}^T \tag{7.12}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\overleftrightarrow{R}(\boldsymbol{n}, \beta) = \sin\beta[\dot{\beta}\boldsymbol{nn} - \dot{\beta}\overleftrightarrow{1} + (\dot{\boldsymbol{n}} \times \overleftrightarrow{1})] + \cos\beta\dot{\beta}(\boldsymbol{n} \times \overleftrightarrow{1})$$
$$+ (1 - \cos\beta)(\dot{\boldsymbol{n}}\boldsymbol{n} + \boldsymbol{n}\dot{\boldsymbol{n}}) \tag{7.13}$$

and

$$\overleftrightarrow{R}^T(\boldsymbol{n}, \beta) = (1 - \cos\beta)\boldsymbol{nn} + \cos\beta\overleftrightarrow{1} - \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) \tag{7.14}$$

The product of $(\mathrm{d}/\mathrm{d}t)\overleftrightarrow{R} \cdot \overleftrightarrow{R}^T$ finally reaches the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\overleftrightarrow{R} \cdot \overleftrightarrow{R}^T = \dot{\beta}\boldsymbol{n} \times \overleftrightarrow{1} + \sin\beta(\dot{\boldsymbol{n}} \times \overleftrightarrow{1}) + (1 - \cos\beta)(\boldsymbol{n} \times \dot{\boldsymbol{n}}) \times \overleftrightarrow{1} \tag{7.15}$$

In the derivation, the vector $\dot{\boldsymbol{n}}$ is assumed to be perpendicular to $\boldsymbol{n}$. The following identities are used for simplification:

$$\boldsymbol{A} \cdot (\boldsymbol{n} \times \overleftrightarrow{1}) = \boldsymbol{A} \times \boldsymbol{n}, \qquad (\boldsymbol{n} \times \overleftrightarrow{1}) \cdot \boldsymbol{A} = \boldsymbol{n} \times \boldsymbol{A} \tag{7.16a}$$

$$(\boldsymbol{A} \times \overleftrightarrow{1}) \cdot (\boldsymbol{B} \times \overleftrightarrow{1}) = \boldsymbol{BA} - \overleftrightarrow{1}(\boldsymbol{A} \cdot \boldsymbol{B}) \tag{7.16b}$$

$$(\dot{\boldsymbol{n}}\boldsymbol{n} - \boldsymbol{n}\dot{\boldsymbol{n}}) = (\boldsymbol{n} \times \dot{\boldsymbol{n}}) \times \overleftrightarrow{1}. \tag{7.16c}$$

Through the use of Eq. (7.12), we obtain

$$\omega = \dot{\beta}\boldsymbol{n} + \sin\beta\dot{\boldsymbol{n}} + (1 - \cos\beta)\boldsymbol{n} \times \dot{\boldsymbol{n}} \tag{7.17}$$

which is the result of this derivation. This is not quite meaningful, however, because $\dot{\boldsymbol{n}}$ is unknown. During the derivation, $\dot{\boldsymbol{n}}$ is assumed to be perpendicular to $\boldsymbol{n}$, but there are many $\dot{\boldsymbol{n}}$ that can satisfy the assumption. Therefore, ω cannot be obtained directly from Eq. (7.17). However, when the rotation is about a fixed axis, Eq. (7.17) reduces to

$$\omega = \dot{\beta}\boldsymbol{n}$$

as expected. On the other hand, from Eq. (7.17) $\dot{\boldsymbol{n}}$ can be expressed in terms of $\boldsymbol{n}$ and ω. The details of derivation are an assigned exercise in the problem section; $\dot{\boldsymbol{n}}$ is obtained as

$$\dot{\boldsymbol{n}} = -\frac{1}{2}\{(\boldsymbol{n} \times \omega) + \cot\frac{\beta}{2}[\boldsymbol{n} \times (\boldsymbol{n} \times \omega)]\} \tag{7.18}$$

Note that from this equation we can see that $\dot{\boldsymbol{n}}$ is perpendicular to $\boldsymbol{n}$ because $\dot{\boldsymbol{n}} \cdot \boldsymbol{n} = 0$, with the components opposite to $(\boldsymbol{n} \times \omega)$ and $\boldsymbol{n} \times (\boldsymbol{n} \times \omega)$.

Although ω cannot be obtained directly from Eq. (7.17), Eq. (7.12) can still lead us to find ω through the rotation operators. In Section 6.7 we have derived the rotation operator with respect to fixed frame of reference as given in Eq. (6.86) for the rotation through Euler angles:

$$\begin{aligned}\overleftrightarrow{R}(\boldsymbol{n}, \beta) &= \overleftrightarrow{R}_3(\boldsymbol{k}'', \psi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \\ &= \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi)\end{aligned}$$

For the operator through the fixed axes, we have

$$\begin{aligned}\omega \times \overleftrightarrow{1} &= \frac{d}{dt}\overleftrightarrow{R} \cdot \overleftrightarrow{R}^T = \frac{d}{dt}[\overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_3] \cdot [\overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_3]^T \\ &= \frac{d}{dt}[\overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_3] \cdot \left[\overleftrightarrow{R}_3^T \cdot \overleftrightarrow{R}_2^T \cdot \overleftrightarrow{R}_1^T\right] \\ &= \frac{d}{dt}\overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_1^T + \overleftrightarrow{R}_1 \cdot \frac{d}{dt}\overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_2^T \cdot \overleftrightarrow{R}_1^T \\ &+ \overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_2 \cdot \frac{d}{dt}\overleftrightarrow{R}_3 \cdot \overleftrightarrow{R}_3^T \cdot \overleftrightarrow{R}_2^T \cdot \overleftrightarrow{R}_1^T \\ &= \omega_\phi \times \overleftrightarrow{1} + \overleftrightarrow{R}_1 \cdot \omega_\theta \times \overleftrightarrow{1} \cdot \overleftrightarrow{R}_1^T \\ &+ \overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_2 \cdot \omega_\psi \times \overleftrightarrow{1} \cdot \overleftrightarrow{R}_2^T \cdot \overleftrightarrow{R}_1^T\end{aligned}$$

where

$$\omega_\phi \times \overleftrightarrow{1} = \frac{d}{dt}\overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_1^T, \qquad \omega_\theta \times \overleftrightarrow{1} = \frac{d}{dt}\overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_2^T, \qquad \omega_\psi \times \overleftrightarrow{1} = \frac{d}{dt}\overleftrightarrow{R}_3 \cdot \overleftrightarrow{R}_3^T$$

have been used. Making use of the identity Eq. (6.76), which is rewritten as follows,

$$\overleftrightarrow{R} \cdot (\boldsymbol{v} \times \overleftrightarrow{1}) \cdot \overleftrightarrow{R}^T = (\overleftrightarrow{R} \cdot \boldsymbol{v}) \times \overleftrightarrow{1}$$

we find

$$\omega = \omega_\phi + \overleftrightarrow{R}_1 \cdot \omega_\theta + \overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_2 \cdot \omega_\psi$$

Note that in this equation

$$\omega_\phi = \dot{\phi}\boldsymbol{k}, \qquad \omega_\theta = \dot{\theta}\boldsymbol{i}, \qquad \omega_\psi = \dot{\psi}\boldsymbol{k}$$

so that

$$\begin{aligned}\omega &= \dot{\phi}\boldsymbol{k} + \overleftrightarrow{R}_1(\boldsymbol{k},\phi)\cdot\dot{\theta}\boldsymbol{i} + \overleftrightarrow{R}_1(\boldsymbol{k},\phi)\cdot\overleftrightarrow{R}_2(\boldsymbol{i},\theta)\cdot\dot{\psi}\boldsymbol{k}\\ &= \dot{\phi}\boldsymbol{k} + \dot{\theta}\boldsymbol{i}' + \dot{\psi}\boldsymbol{k}''\end{aligned} \tag{7.19}$$

which is certainly true. The ω can be expressed in any frame of reference. Rewriting the operators in Eq. (7.19) in detail, we have, in the fixed frame of XYZ,

$$\begin{aligned}\omega &= \dot{\phi}\boldsymbol{k} + [\boldsymbol{kk} + \cos\phi(\boldsymbol{ii}+\boldsymbol{jj}) + \sin\phi(\boldsymbol{k}\times\overleftrightarrow{1})]\cdot\dot{\theta}\boldsymbol{i}\\ &\quad + \overleftrightarrow{R}_1(\boldsymbol{k}\phi)\cdot[(1-\cos\theta)\boldsymbol{ii} + \cos\theta\overleftrightarrow{1} + \sin\theta(\boldsymbol{i}\times\overleftrightarrow{1})]\cdot\dot{\psi}\boldsymbol{k}\\ &= \dot{\phi}\boldsymbol{k} + \dot{\theta}(\cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j}) + \overleftrightarrow{R}_1(\boldsymbol{k},\phi)\cdot[\cos\theta\boldsymbol{k} - \sin\theta\boldsymbol{j}]\dot{\psi}\\ &= \boldsymbol{i}(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi) + \boldsymbol{j}(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi)\\ &\quad + \boldsymbol{k}(\dot{\phi} + \dot{\psi}\cos\theta)\end{aligned} \tag{7.20}$$

This result agrees well with Eq. (7.10), which was derived through matrix operations. In the rotation of axes for the Euler angles, because $\boldsymbol{i}' = \boldsymbol{i}''$, $\boldsymbol{k} = \boldsymbol{k}'$, the expression in Eq. (7.19) can be converted easily into the double-primed frame. Through the use of $\boldsymbol{k}' = \overleftrightarrow{R}_2^T(\boldsymbol{i}'',\theta)\cdot\boldsymbol{k}''$, we obtain

$$\omega = \boldsymbol{i}''\dot{\theta} + \boldsymbol{j}''(\dot{\phi}\sin\theta) + \boldsymbol{k}''(\dot{\psi} + \dot{\phi}\cos\theta) \tag{7.21}$$

This can be converted into the triple-primed frame with the use of

$$\boldsymbol{i}'' = \overleftrightarrow{R}_3^T(\boldsymbol{k}''',\psi)\cdot\boldsymbol{i}''', \qquad \boldsymbol{j}'' = \overleftrightarrow{R}_3^T(\boldsymbol{k}''',\psi)\cdot\boldsymbol{j}''', \qquad \boldsymbol{k}'' = \boldsymbol{k}'''$$

In the triple-primed frame of reference, we find

$$\begin{aligned}\omega &= \boldsymbol{i}'''(\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi) + \boldsymbol{j}'''(-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi)\\ &\quad + \boldsymbol{k}'''(\dot{\psi} + \dot{\phi}\cos\theta)\end{aligned} \tag{7.22}$$

Euler Equations of Motion

In an inertial frame of reference, such as the set of axes XYZ, for $\boldsymbol{G} = \boldsymbol{L} =$ angular momentum of a body, Eq. (7.1) gives

$$\left(\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right)_{XYZ} = \left(\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right)_{xyz} + \omega\times\boldsymbol{L} = \boldsymbol{N}$$

where the xyz frame is in rotational motion with velocity ω relative to the XYZ frame, and $\boldsymbol{N}$ is the torque applied to the body. In general, as given in Eq. (6.37) and Eq. (6.39)

$$L = \overleftrightarrow{I}_m \cdot \omega$$

We can simplify this expression by rotating the axes of xyz to coincide with the principal axes of the body and label them $x'y'z'$. Then we have

$$L_{x'} = I_1\omega_{x'}, \qquad L_{y'} = I_2\omega_{y'}, \qquad L_{z'} = I_3\omega_{z'}$$

The full set of Euler equations are reduced to

$$N_{x'} = I_1\dot{\omega}_{x'} - \omega_{y'}\omega_{z'}(I_2 - I_3) \tag{7.23a}$$

$$N_{y'} = I_2\dot{\omega}_{y'} - \omega_{z'}\omega_{x'}(I_3 - I_1) \tag{7.23b}$$

$$N_{z'} = I_3\dot{\omega}_{z'} - \omega_{x'}\omega_{y'}(I_1 - I_2) \tag{7.23c}$$

Let us first consider Euler's angular velocity for the preceding equations. Note that the triple-primed axes are actually the axes fixed in the rotating body, so that ω given in Eq. (7.22) is to be used for this set of equations:

$$\omega_{x'} = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi$$

$$\omega_{y'} = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi$$

$$\omega_{z'} = \dot{\psi} + \dot{\phi}\cos\theta$$

Differentiating with respect to time and substituting into Eqs. (7.23a–7.23c) gives

$$\begin{aligned} N_{x'} = {} & I_1[\ddot{\theta}\cos\psi + \ddot{\phi}\sin\theta\sin\psi] + (I_1 - I_2)[-\dot{\theta}\dot{\psi}\sin\psi + \dot{\phi}\dot{\psi}\sin\theta\cos\psi] \\ & + (I_1 + I_2)\dot{\phi}\dot{\theta}\cos\theta\sin\psi + I_3[-\dot{\theta}\dot{\psi}\sin\psi + \dot{\phi}\dot{\psi}\sin\theta\cos\psi \\ & - \dot{\theta}\dot{\phi}\cos\theta\sin\psi + \dot{\phi}^2\sin\theta\cos\theta\cos\psi] \end{aligned} \tag{7.24a}$$

$$\begin{aligned} N_{y'} = {} & I_2(-\ddot{\theta}\sin\psi + \ddot{\phi}\sin\theta\cos\psi) + (I_1 - I_2)[\dot{\theta}\dot{\psi}\cos\psi + \dot{\phi}\dot{\psi}\sin\theta\sin\psi] \\ & + (I_1 + I_2)\dot{\theta}\dot{\phi}\cos\theta\cos\psi - I_3[\dot{\theta}\dot{\psi}\cos\psi + \dot{\theta}\dot{\phi}\cos\theta\cos\psi \\ & + \dot{\phi}\dot{\psi}\sin\theta\sin\psi + \dot{\phi}^2\cos\theta\sin\theta\sin\psi] \end{aligned} \tag{7.24b}$$

$$\begin{aligned} N_{z'} = {} & I_3[\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta] - (I_2 - I_1)[-\dot{\theta}^2\cos\psi\sin\psi \\ & + \dot{\theta}\dot{\phi}\sin\theta\cos 2\psi + \dot{\phi}^2\sin^2\theta\cos\psi\sin\psi] \end{aligned} \tag{7.24c}$$

The application of the preceding equations is demonstrated in the following example.

Example 7.1

A toy gyroscopic top is shown in Fig. 7.4. The gravitational force on the disk is W. If the disk is given a high angular velocity ω_s about its shaft oz' and one end of the shaft is placed on a pedestal, it is observed that the shaft and disk will not fall but will precess around the axis oZ because of torque $W\ell$ acting on the system. Find the angular velocity for precession.

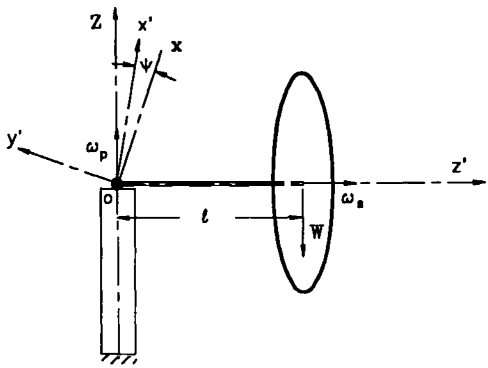

Fig. 7.4 Toy gyroscopic top.

Solution. Applying Eqs. (7.24a–7.24c) to this problem, we make some necessary assumptions and have

$$I_1 = I_2, \qquad I_3 = I$$

$$\theta = 90\,\text{deg}, \qquad \dot{\theta} = 0, \qquad \ddot{\theta} = \ddot{\phi} = \ddot{\psi} = 0, \qquad \dot{\psi} = \omega_s, \qquad \dot{\phi} = \omega_p$$

$$N_{x'} = W\ell\cos\psi, \qquad N_{y'} = -W\ell\sin\psi, \qquad N_{z'} = 0$$

Note that the torque produced by the weight is in the direction of x, and axes x, x', and y' are in the same plane. The x', y', and z' axes are embedded in the rotating top. Either from Eq. (7.24a) or (7.24b), we find

$$W\ell = I\dot{\phi}\dot{\psi} = I\omega_s\omega_p, \qquad \omega_p = W\ell/I\omega_s$$

Therefore, the angular velocity for precession is directly proportional to the torque produced by its own weight and inversely proportional to the angular momentum along the spinning axis.

7.4 Gyroscopic Motion

To study the motion of a gyroscope, it is convenient to consider the rotating body and the rotating coordinate system separately. Let the coordinate axes lie along the principal axes of the body but allow the body to spin in the rotating coordinate system with a rotating velocity of $\dot{\psi}$ along the z'' axis, as shown in Fig. 7.5. Hence,

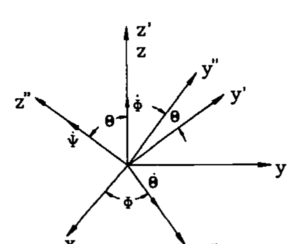

Fig. 7.5 Euler's angular velocities.

the angular velocity of the rotating frame of reference is

$$\Omega = \dot{\theta} i'' + \dot{\phi} \sin\theta j'' + \dot{\phi} \cos\theta k'' \tag{7.25}$$

and the angular velocity of the body is

$$\omega = \dot{\theta} i'' + \dot{\phi} \sin\theta j'' + (\dot{\psi} + \dot{\phi} \cos\theta) k'' \tag{7.26}$$

in which $\dot{\psi}$ is the angular velocity of the spin, $\dot{\phi}$ is the angular velocity of precession, and $\dot{\theta}$ is the angular velocity of nutation.

It is also assumed that there is always one point in the system that is fixed either in a fixed system or in an inertial frame of reference. This point may be the center of mass or one of the supports. Applying the Euler equations, we find

$$\left(\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right)_{XYZ} = \left(\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right)_{xyz} + \boldsymbol{\Omega} \times \boldsymbol{L} = \boldsymbol{N} \tag{7.27}$$

where $L_1 = I'\omega_1, L_2 = I'\omega_2, L_3 = I\omega_3$; I' is the mass moment of inertia with respect to the x or y axis; and I is the mass moment of inertia with respect to the z axis.

Substituting the expressions for Ω and ω in Eqs. (7.25) and (7.26) into Eq. (7.27) leads to

$$N_1 = I'\ddot{\theta} + (I - I')(\dot{\phi}^2 \sin\theta \cos\theta) + I\dot{\phi}\dot{\psi} \sin\theta \tag{7.28a}$$

$$N_2 = I'\ddot{\phi} \sin\theta + 2I'\dot{\theta}\dot{\phi} \cos\theta - I(\dot{\psi} + \dot{\phi} \cos\theta)\dot{\theta} \tag{7.28b}$$

$$N_3 = I(\ddot{\psi} + \ddot{\phi} \cos\theta - \dot{\phi}\dot{\theta} \sin\theta) \tag{7.28c}$$

Example 7.2

In Fig. 7.6, the propeller shaft of an airplane is shown. The propeller rotates at 2000 rpm clockwise (cw) when viewed from the rear and is driven by the engine through reduction gears. Suppose the airplane flies horizontally and makes a turn to the right at 0.2 rad/s as viewed from above. The propeller has a mass of 30 kg and moment of inertia of 25 kg-m^2. Find the gyroscopic forces that the propeller shaft exerts against bearings A and B, which are 150 mm apart.

Solution. To simplify the problem, it is assumed that, before the airplane begins making a turn, the whole system is in an inertial frame of reference for Eqs. (7.28a–7.28c) to apply. The angular momentum of the propeller is in the direction of z'' and is

$$\dot{\psi} = \omega_3 = \frac{2000(2\pi)}{60} = 209 \text{ rad/s}$$

$$L_3 = I\dot{\psi} = 25 \times 209 = 5225 \text{ kg-m}^2/\text{s}$$

$$\dot{\phi} = -0.2 \text{ rad/s}$$

$$\theta = 90 \text{ deg}, \qquad \dot{\theta} = \ddot{\theta} = \ddot{\phi} = \ddot{\psi} = 0$$

$$N_1 = I\dot{\psi}\dot{\phi} = -25(209)(0.2) = -1045 \text{ N-m}$$

$$F = \frac{|N_1|}{\ell} = \frac{+1045}{0.150} = 6967 \text{ N}$$

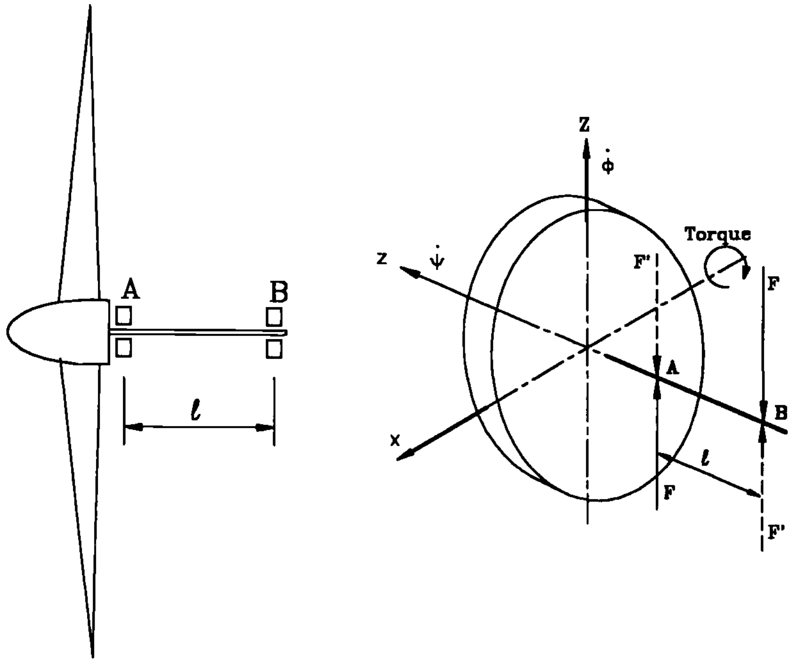

Fig. 7.6 Gyroscopic effect on propeller shaft.

Note that the force acting on the bearing F' is in the opposite direction of F. From the couple formed by F', we can see that the moment produced by F' causes the airplane to dive. On the other hand, if the airplane is turning to the left, then the moment from the bearings pitches the airplane upward.

Example 7.3

Shown in Fig. 7.7 is a single-degree-of-freedom gyro. The spin axis of disc E is held by a gimbal A that can rotate about bearings C and D. These bearings are supported by the gyro case which, in turn, is clamped to the vehicle to be guided. If the gyro case rotates about a vertical axis while the rotor is spinning about the horizontal axis, then the gimbal A will tend to rotate about CD in an attempt to align with the vertical. When gimbal A is restrained by a set of springs S with a combined torsional spring constant given as k_t, then the gyro is called a rate gyro. The neutral position of the springs is set at $\theta = \pi/2$. If the rotation of the gyro case is constant and the gimbal A assumes a fixed orientation relative to the vertical as a result of the restraining springs, we have a case of regular precession. The rotation of the gyro case gives the precession speed $\dot{\phi}$ about the precession axis, which is clearly the vertical axis. The nutation angle θ is then the orientation of gimbal A (i.e., the z axis) with respect to the Z axis.

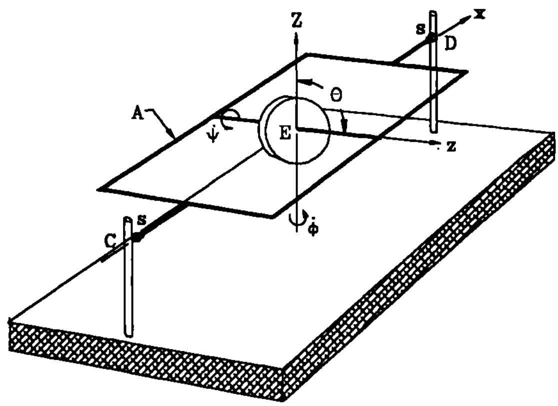

Fig. 7.7 Single-degree-of-freedom gyro.

Given the following data, what is θ for the condition of steady precession?

$I = 4 \times 10^{-4}$ kg-m^2
$I' = 2 \times 10^{-4}$ kg-m^2
$\dot{\psi} = 20{,}000$ rad/s
$k_t = 1.0$ N-m/rad
$\dot{\phi} = 0.1$ rad/s

Solution. Taking the x axis along CD, the z axis along the spinning axis of the rotor, and the Z axis for the precesion axis, we have, from Eq. (7.28a),

$$N_1 = k_t(\pi/2 - \theta) = (I - I')(\dot{\phi}^2 \sin\theta \cos\theta) + I\dot{\phi}\dot{\psi} \sin\theta$$

$$(\pi/2 - \theta) = 2 \times 10^{-4}[(0.1)^2 \sin\theta \cos\theta] + 4 \times 10^{-4}(0.1)(20{,}000) \sin\theta = (2 \times 10^{-6} \cos\theta + 0.8) \sin\theta$$

Neglecting $2 \times 10^{-6} \cos\theta$, which is much smaller than 0.8, the equation becomes

$$\pi/2 - \theta = 0.8 \sin\theta$$

$$\theta = 53 \text{ deg}$$

In practice, the torque N_1 is measured. Because N_1 and the rotating velocity of the vehicle $\dot{\phi}$ are directly related, the required value of $\dot{\phi}$ can be calculated from the measured value of N_1.

Example 7.4

We shall now explain the effect of the Earth's rotation on the operation of the gyro-compass. The gyro-compass is a two-degree-of-freedom gyroscope as shown in Fig. 7.8a with torsional springs restricting the x axis. This device gives the direction to the geometric north pole (not the magnetic north pole) if it is set to that direction at the beginning of observation. In this example we will see that the Earth's rotation causes some oscillation of the spinning axis about the meridian.

For simplicity, we consider a gyro-compass at a fixed position on the Earth's surface. The body axis z of the gyro-compass can rotate in plane T tangent to the Earth's surface as shown in Fig. 7.8b, where the z axis is at an angle α with the tangent to the meridian line. Because the angle α may vary with time, there is a possible angular velocity $\dot{\alpha}$ normal to the plane T. The y axis is a radial line from the center of Earth at o, and, therefore, is always collinear with $\dot{\alpha}$. The x axis then is chosen to form a right-hand triad and is in plane T. An inertial reference XYZ is chosen at the center of the Earth so that the Z axis is along the north–south axis. The gyroscope rotates with spin velocity $\dot{\psi}$ along z and swinging velocity $\dot{\alpha}$ along y and precession velocity $\dot{\phi}$ along Z, where $\dot{\phi}$ is the angular velocity of the Earth, a constant vector of small magnitude. For convenience, another Z axis has been set up at the gyroscope. The angle between the Z axis and the tangent to the meridian designated as λ is the latitude of the position of the gyro-compass. Note that the nutation velocity $\dot{\theta}$ is not used here. It is a function of α, λ, and $\dot{\alpha}$.

Because the axes xyz are not fixed to the body, we must use Eq. (7.27) for the equation of motion. We have

$$L_1 = I'\omega_1, \qquad L_2 = I'\omega_2, \qquad L_3 = I\omega_3$$

and

$$\omega_1 = -\dot{\phi}\cos\lambda\sin\alpha$$

$$\omega_2 = \dot{\alpha} + \dot{\phi}\sin\lambda$$

$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\lambda\cos\alpha$$

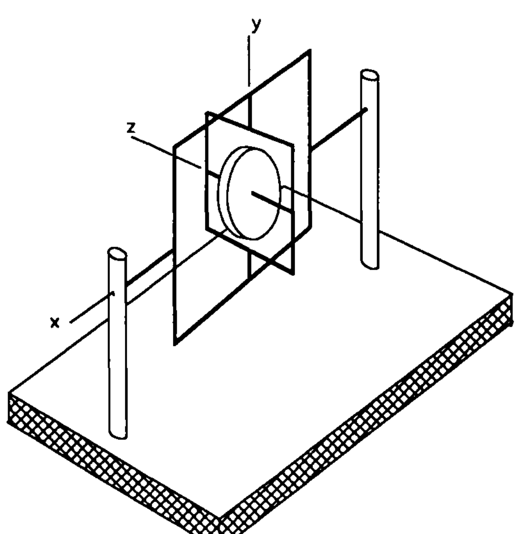

Fig. 7.8a Two-degree-of-freedom gyro (rotation about x axis restricted).

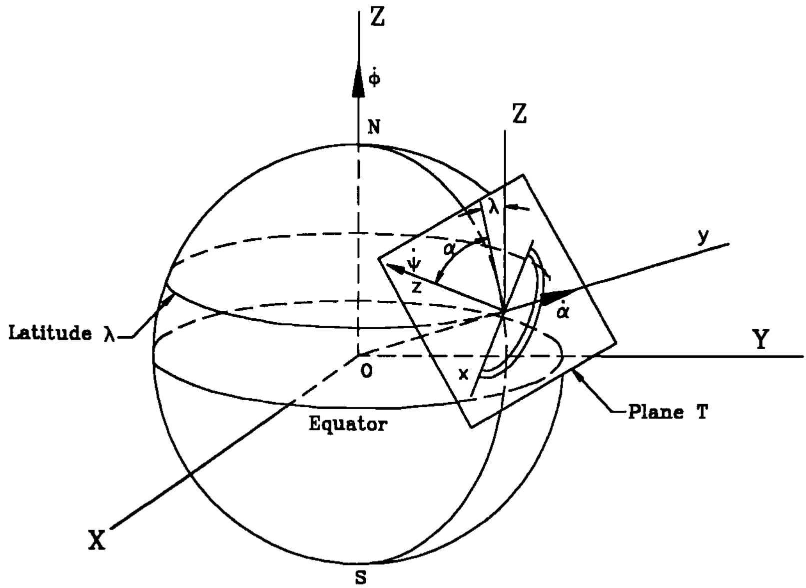

Fig. 7.8b Gyro-compass in oscillation.

The angular velocity components of xyz are

$$\Omega_1 = -\dot{\phi}\cos\lambda\sin\alpha$$

$$\Omega_2 = \dot{\alpha} + \dot{\phi}\sin\lambda$$

$$\Omega_3 = \dot{\phi}\cos\lambda\cos\alpha$$

Now we can write Eq. (7.27) as

$$\begin{aligned} N_1\boldsymbol{i} + N_2\boldsymbol{j} + N_3\boldsymbol{k} = {} & I'\frac{\mathrm{d}}{\mathrm{d}t}(-\dot{\phi}\cos\alpha\sin\alpha)\boldsymbol{i} \\ & + I'\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\alpha} + \dot{\phi}\sin\lambda)\boldsymbol{j} + I\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\psi} + \dot{\phi}\cos\lambda\cos\alpha)\boldsymbol{k} \\ & + I'(\dot{\alpha} + \dot{\phi}\sin\lambda)[(-\dot{\phi}\cos\lambda\sin\alpha)\boldsymbol{k} - \dot{\phi}\cos\lambda\cos\alpha\boldsymbol{i}] \\ & + I'(-\dot{\phi}\cos\lambda\sin\alpha)[\dot{\phi}\cos\lambda\cos\alpha\boldsymbol{j} - (\dot{\alpha} + \dot{\phi}\sin\lambda)\boldsymbol{k}] \\ & + I(\dot{\psi} + \dot{\phi}\cos\lambda\cos\alpha)[(\dot{\alpha} + \dot{\phi}\sin\lambda)\boldsymbol{i} + (\dot{\phi}\cos\lambda\sin\alpha)\boldsymbol{j}] \end{aligned}$$

Dividing the preceding equation into three components, we find

$$\begin{aligned} N_1 = {} & I'[(-\dot{\phi}\dot{\alpha}\cos\lambda\cos\alpha) + (\dot{\alpha} + \dot{\phi}\sin\lambda)(-\dot{\phi}\cos\lambda\cos\alpha)] \\ & + I(\dot{\psi} + \dot{\phi}\cos\lambda\cos\alpha)(\dot{\alpha} + \dot{\phi}\sin\lambda) \end{aligned} \tag{7.29a}$$

$$N_2 = I'(\ddot{\alpha} - \dot{\phi}^2\cos^2\lambda\sin\alpha\cos\alpha) + I(\dot{\psi} + \dot{\phi}\cos\lambda\cos\alpha)\dot{\phi}\cos\lambda\sin\alpha \tag{7.29b}$$

$$N_3 = I(\ddot{\psi} - \dot{\phi}\dot{\alpha}\cos\lambda\sin\alpha) \tag{7.29c}$$

Now let us consider the external torques acting on the gyro-compass system. Because the spin axis is kept in the plane T, a proper amount of N_1 must be applied along the x axis. There are no torques along y and z axes, i.e., $N_2 = N_3 = 0$, and since $\dot{\phi}$ is expected to be much smaller than $\dot{\psi}$, and $\dot{\phi}^2 \ll \ddot{\alpha}$, Eqs. (7.29b) and (7.29c) are reduced to

$$I'\ddot{\alpha} + I\dot{\psi}\dot{\phi}\cos\lambda\sin\alpha = 0 \tag{7.30a}$$

$$I(\ddot{\psi} - \dot{\phi}\dot{\alpha}\cos\lambda\sin\alpha) = 0 \tag{7.30b}$$

Note that $\dot{\psi}$ is the spin velocity of the rotor, $\dot{\psi} \gg \dot{\phi}$, and $\dot{\psi} \gg \dot{\alpha}$. Equation (7.30b) may be approximated as $\ddot{\psi} = 0$, i.e., as $\dot{\psi}$ is a constant. Then Eq. (7.30a) can be written in the form of

$$\ddot{\alpha} + c\alpha = 0 \tag{7.31}$$

where

$$c = \frac{I\dot{\psi}\dot{\phi}\cos\lambda}{I'}$$

We also assume that α is much less than one. Equation (7.31) means that the Earth's rotation can cause the spin axis to oscillate about the meridian. The frequency of oscillation is

$$f = \frac{1}{2\pi}\sqrt{\frac{I\dot{\psi}\dot{\phi}\cos\lambda}{I'}} \tag{7.32}$$

Plugging realistic values into this equation, let $\dot{\psi} = 20{,}000$ rad/s, $\dot{\phi} = 7.2722 \times 10^{-5}$ rad/s, $\lambda = 20$ deg, and $I = 2I'$, then we find

$$f = 0.263 \text{ cycle/s}$$

or the period of oscillation is 3.8 s.

7.5 Motion of a Heavy Symmetrical Top

The motion of a rotating top is well known and is a good example to learn how powerful mathematical techniques are used to extract a great deal of physical information using minimal calculation. Nutation and precession will be studied in detail.

We choose the symmetry axis of the top as the z axis and fix the supporting point of the top at the origin of coordinates. The center of mass is located on the z axis at distance ℓ from the origin as shown in Fig. 7.9. The Euler angles were originally designed for the treatment of a rotating top, and they will prove to be very convenient. To find the equations of motion, Lagrangian techniques are applied because they are simpler than the Euler equations. With this in mind, we write the Lagrangian function as

$$L = T - V = \tfrac{1}{2}I'\left(\omega_x^2 + \omega_y^2\right) + \tfrac{1}{2}I\omega_z^2 - Mg\ell\cos\theta \tag{7.33}$$

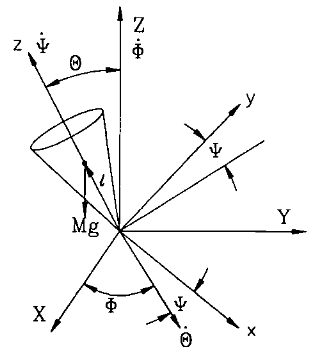

Fig. 7.9 Coordinates for the heavy symmetrical top.

in which $I_1 = I_2 = I'$ because of symmetry and $I_3 = I$ have been used. Clearly, the angular velocity of the top expressed in the body axes is most convenient. From Eq. (7.22) we have

$$\boldsymbol{\omega} = (\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\boldsymbol{i} + (-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi)\boldsymbol{j} + (\dot{\psi} + \dot{\phi}\cos\theta)\boldsymbol{k} \tag{7.34}$$

With the use of Eq. (7.34), the Lagrangian function, Eq. (7.33) becomes

$$L = \tfrac{1}{2}I'(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \tfrac{1}{2}I(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mg\ell\cos\theta \tag{7.35}$$

As discussed in Chapter 4, ψ and ϕ are ignorable coordinates because they do not appear in the Lagrangian function. Consequently, the two angular momenta P_ψ and P_ϕ are constant, i.e.,

$$P_\psi \equiv \frac{\partial L}{\partial \dot{\psi}} \equiv I(\dot{\psi} + \dot{\phi}\cos\theta) = I\omega_z = \text{const} \tag{7.36}$$

$$\begin{aligned} P_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} &= I'(\sin^2\theta)\dot{\phi} + I(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta \\ &= (I'\sin^2\theta + I\cos^2\theta)\dot{\phi} + I\dot{\psi}\cos\theta = \text{const} \end{aligned} \tag{7.37}$$

Furthermore, because no frictional dissipation is assumed in this analysis, the total energy $E = T + V$ is constant. In view of Eq. (7.36), subtraction of E by $\frac{1}{2}I\omega_z^2$ is still constant.

$$E' = E - \tfrac{1}{2}I\omega_z^2 = \tfrac{1}{2}I'(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + Mg\ell\cos\theta = \text{const}$$

From Eq. (7.36) we have

$$\dot{\psi} = \omega_z - \dot{\phi}\cos\theta \tag{7.38}$$

Substitution of $\dot{\psi}$ into Eq. (7.37) gives

$$\begin{aligned} &(I'\sin^2\theta + I\cos^2\theta)\dot{\phi} + I\cos\theta(\omega_z - \dot{\phi}\cos\theta) \\ &= I'(\sin^2\theta)\dot{\phi} + I\omega_z\cos\theta = \text{const} = I'B \end{aligned}$$

Rearranging, we find

$$\dot{\phi} = \frac{B - A\cos\theta}{\sin^2\theta} \tag{7.39}$$

where $A = (I/I')\omega_z$.

Then, from Eq. (7.38), we have

$$\dot{\psi} = \omega_z - \frac{\cos\theta}{\sin^2\theta}(B - A\cos\theta) \tag{7.40}$$

Substituting Eq. (7.39) into the expression for energy E' gives

$$E' = \frac{1}{2}I'\left[\dot{\theta}^2 + \frac{(B - A\cos\theta)^2}{\sin^2\theta}\right] + Mg\ell\cos\theta = \text{const}$$

or

$$(\sin^2\theta)\dot{\theta}^2 = (C - D\cos\theta)\sin^2\theta - (B - A\cos\theta)^2 \tag{7.41}$$

where

$$C = \frac{2E'}{I'}, \qquad D = 2\frac{Mg\ell}{I'}$$

Equation (7.41) is a first-order differential equation. The nutational motion of the rotating shaft can be predicted from this equation. Having found the function $\theta(t)$, the precession of the top can be obtained through Eq. (7.39), and the variation of spinning velocity can be found from Eq. (7.40). Equation. (7.41) is a nonlinear equation, however, which cannot be integrated analytically. Much information may be obtained without integration of these equations. Let us change the variable in the equation with

$$\mu = \cos\theta$$

Then we have

$$\dot{\mu}^2 = (C - D\mu)(1 - \mu^2) - (B - A\mu)^2 = f(\mu)$$

The result $\theta(t)$ of the preceding equation depends highly on the behavior of the function $f(\mu)$. By introducing proper numerical values for A, B, C, and D, the variations of $f(\mu)$ are obtained as shown in Fig. 7.10. It easily is seen that there are three roots. From the plot, two roots are between 0 and 1 and are reasonable roots because $0 \leq \cos\theta \leq 1$; the third root is impossible. Note that at those two roots $\dot{\mu} = 0$, i.e., $\dot{\theta} = 0$, so that θ reaches minimum or maximum at these roots; also note that $f(\mu)$ is positive between these two roots so that $\dot{\mu} = \pm\sqrt{f(\mu)}$ or $d\mu$ can be positive and negative. With this understanding, Eqs. (7.41) and (7.40) are numerically integrated. Three different possible cases are given in Figs. 7.11. For

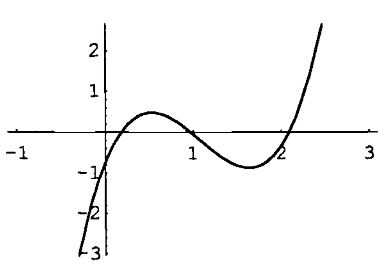

Fig. 7.10 Plot of $f(\mu)$ with A = 2, B = 1.8, C = 2.5, and D = 2.

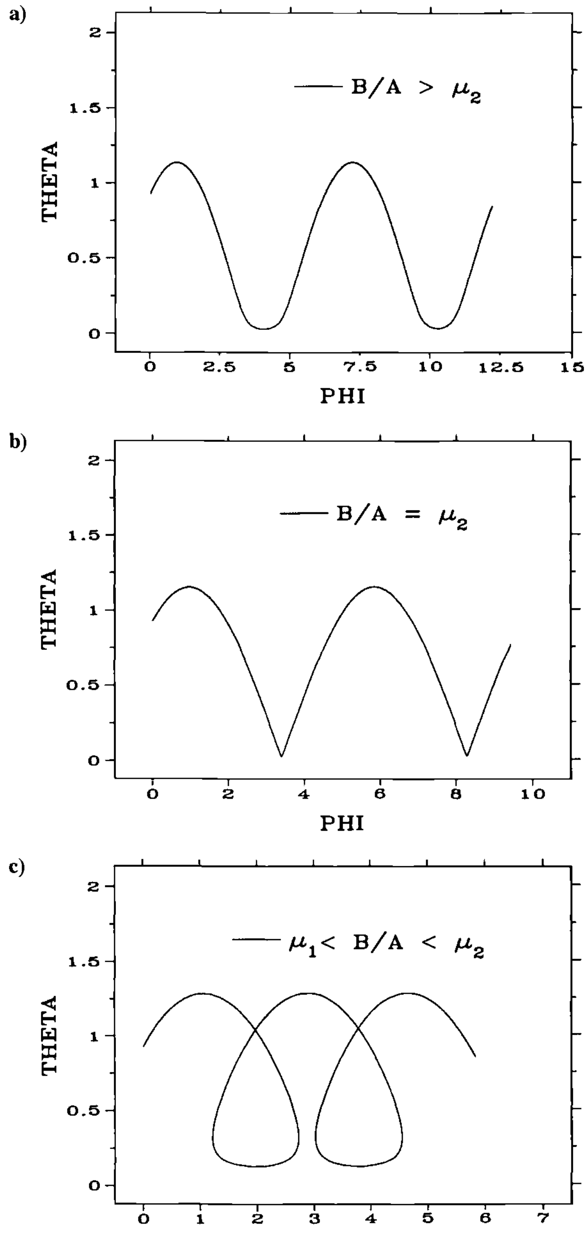

Fig. 7.11 Three different possible nutations.

Fig. 7.11a, where the value of B/A is greater than μ_2, the motion is called regular precession with nutation because precession occurs at nearly constant speed. For Fig. 7.11b, where the value of $B/A = \mu_2, \dot{\theta} = 0$, and $\dot{\phi} = 0$ as $\mu = \mu_2$, so that cusps are shown at $\theta_{\min}$. For Fig. 7.11c, the value of B/A is between the first two roots, so that $\dot{\phi}$ can be positive and negative. Consequently, loops are shown in this case. In the numerical integration, because the two integral limits are the roots in the denominator of the integrand, Simpson's one-third rule with $d\mu = 0.0001$ is employed for integration and with a further reduced interval near the integral limits.

7.6 Torque on a Satellite in Circular Orbit

During the last four decades, we have launched many objects into space and have encountered many engineering problems specific to motion in orbit. As the motion of airplanes was well studied in the beginning of the 20th century, the motion of the space station moving in orbit now requires diligent study so that some induced motions during flight operations can be anticipated and delicate space vehicles are designed to endure the additional stresses they may encounter. Certainly there are many possible ways to analyze the problem. The following approach was first given by E. Neal Moore.*

Consider a satellite moving in a circular orbit around Earth. The coordinate system xyz is so chosen that the z axis is from the center of Earth pointing outward through the center of mass of the satellite. A plane contains the z axis and the orbit curve is called the orbit plane. The y axis is in the orbit plane. The angular velocity ω of the satellite relative to the Earth is perpendicular to that plane. The x axis is antiparallel to ω. The origin of the x, y, z coordinates is chosen at the center of mass of the satellite. The body of the satellite is not fixed in the xyz system so that it can pitch, roll, and yaw relative to the axes of xyz system. With the coordinate system chosen, now let us consider that a small element dm as shown in Fig. 7.12 and consider that the frame of reference in the Earth is the inertial frame of reference.

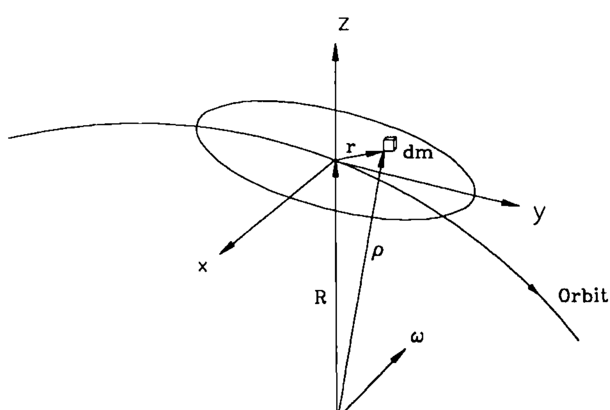

Fig. 7.12 Satellite in a circular orbit.

*Moore, E. N., *Theoretical Mechanics*, Wiley, New York, 1983, Chap. 6.

Applying Eq. (7.5) with the gravitational force applied to the satellite as the only external force, we have

$$\mathrm{d}\boldsymbol{F} = dm\boldsymbol{a} = -GMdm\frac{\boldsymbol{\rho}}{\rho^3} - dm\ddot{\boldsymbol{R}} - dm\dot{\boldsymbol{\omega}} \times \boldsymbol{r} - dm\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) - 2dm(\boldsymbol{\omega} \times \boldsymbol{v}) \qquad (7.42)$$

where $\boldsymbol{v}$ is the velocity of dm as observed in the xyz system. As the body is rotating relative to the moving coordinate system with angular velocity $\boldsymbol{\omega}'$, then

$$\boldsymbol{v} = \boldsymbol{\omega}' \times \boldsymbol{r}$$

where $\boldsymbol{r}$ is the position vector of dm. The torque acting on the body of the satellite about the center of mass because of its own motion is obtained by integration of torque over the whole body

$$\boldsymbol{N} = \int_{\text{boby}} \boldsymbol{r} \times \mathrm{d}\boldsymbol{F} \qquad (7.43)$$

Making use of the fact that $\boldsymbol{R} = R\boldsymbol{K}$

$$\dot{\boldsymbol{R}} = R\dot{\boldsymbol{K}} = R\boldsymbol{\omega} \times \boldsymbol{K} = \boldsymbol{\omega} \times \boldsymbol{R}$$

$$\ddot{\boldsymbol{R}} = \boldsymbol{\omega} \times \dot{\boldsymbol{R}} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{R}), \qquad \boldsymbol{\omega} = \text{const}$$

In addition, $\boldsymbol{\rho} = \boldsymbol{R} + \boldsymbol{r}$. Substituting these expressions into Eq. (7.42) leads to

$$\mathrm{d}\boldsymbol{F} = -GMdm\frac{\boldsymbol{\rho}}{\rho^3} - dm\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) - 2dm(\boldsymbol{\omega} \times \boldsymbol{v}) \qquad (7.44)$$

The first term on the right-hand side of the preceding equation is called the gravity term; the second term is the centrifugal term, and the third term is the Coriolis term. They are to be examined separately as follows.

1) For the gravity term, because

$$\rho^2 = R^2 + r^2 + 2\boldsymbol{R} \cdot \boldsymbol{r}$$

$$\rho^3 = [R^2 + r^2 + 2\boldsymbol{R} \cdot \boldsymbol{r}]^{\frac{3}{2}} = R^3\left[1 + \left(\frac{r}{R}\right)^2 + 2\frac{\boldsymbol{R} \cdot \boldsymbol{r}}{R^2}\right]^{\frac{3}{2}}$$

$$\frac{1}{\rho^3} = \frac{1}{R^3}\left[1 + \left(\frac{r}{R}\right)^2 + 2\frac{\boldsymbol{R} \cdot \boldsymbol{r}}{R^2}\right]^{-\frac{3}{2}}$$

$$\cong \frac{1}{R^3}\left[1 - \frac{3\boldsymbol{R} \cdot \boldsymbol{r}}{R^2}\right] \qquad \text{for } R \gg r$$

and because the satellite is in circular orbit

$$\frac{GMm}{R^2} = mR\omega^2$$

$$\frac{GM}{R^3} = \omega^2$$

The torque produced by the gravitational effect is found to be

$$\begin{aligned} \boldsymbol{N}_g &= -\int GMdm\frac{\boldsymbol{r}\times\boldsymbol{\rho}}{\rho^3} \\ &= -\omega^2\int dm(\boldsymbol{r}\times\boldsymbol{\rho})\left(1-\frac{3\boldsymbol{R}\cdot\boldsymbol{r}}{R^2}\right) \end{aligned}$$

With the use of $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $\boldsymbol{R} = R\boldsymbol{k}$, the equation is simplified to

$$\begin{aligned} \boldsymbol{N}_g &= -\omega^2\int dm(\boldsymbol{r}\times\boldsymbol{R})\left(1-\frac{3z}{R}\right) = -\omega^2\int dmR(-x\boldsymbol{j}+y\boldsymbol{i})\left(1-\frac{3z}{R}\right) \\ &= 3\omega^2\int dmz(-x\boldsymbol{j}+y\boldsymbol{i}) = 3\omega^2(-I_{yz}\boldsymbol{i}+I_{xz}\boldsymbol{j}) \end{aligned} \tag{7.45}$$

where

$$I_{yz} = -\int zydm \tag{7.46a}$$

$$I_{xz} = -\int xzdm \tag{7.46b}$$

2) For the centrifugal term, because

$$\boldsymbol{\omega} = -\omega\boldsymbol{i}$$

$$\boldsymbol{N}_{\text{cen}} = -\int dm\boldsymbol{r}\times[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})] = -\omega^2\int dm\boldsymbol{r}\times[\boldsymbol{i}\times(\boldsymbol{i}\times\boldsymbol{\rho})]$$

Now, because

$$\boldsymbol{i}\times(\boldsymbol{i}\times\boldsymbol{\rho}) = \boldsymbol{i}\times[\boldsymbol{i}\times(R\boldsymbol{k}+x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})] = -y\boldsymbol{j}-(R+z)\boldsymbol{k}$$

we find

$$\begin{aligned} \boldsymbol{N}_{\text{cen}} &= -\omega^2\int dm\boldsymbol{r}\times[y\boldsymbol{j}+(R+z)\boldsymbol{k}] = \omega^2\int dm(xy\boldsymbol{k}-xz\boldsymbol{j}) \\ &= \omega^2(I_{xz}\boldsymbol{j}-I_{xy}\boldsymbol{k}) \end{aligned} \tag{7.47}$$

where

$$I_{xz} = -\int xzdm \tag{7.48a}$$

$$I_{xy} = -\int xydm \tag{7.48b}$$

3) The Coriolis term is

$$N_{\text{cor}} = -2\int dm\boldsymbol{r} \times (\omega \times \boldsymbol{v})$$

Because

$$\boldsymbol{v} = \omega' \times \boldsymbol{r} = (\omega'_x\boldsymbol{i} + \omega'_y\boldsymbol{j} + \omega'_z\boldsymbol{k}) \times (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$$

$$= \omega'_x y\boldsymbol{k} - \omega'_x z\boldsymbol{j} - \omega'_y x\boldsymbol{k} + \omega'_y z\boldsymbol{i} + \omega'_z x\boldsymbol{j} - \omega'_z y\boldsymbol{i}$$

$$\omega \times \boldsymbol{v} = -\boldsymbol{i}\omega \times \boldsymbol{v} = \omega(\omega'_x y - \omega'_y x)\boldsymbol{j} + \omega(\omega'_x z - \omega'_z x)\boldsymbol{k}$$

$$\boldsymbol{r} \times (\omega \times \boldsymbol{v}) = \omega(\omega'_y xz - \omega'_z xy)\boldsymbol{i} + \omega(-\omega'_x xz + \omega'_z x^2)\boldsymbol{j}$$

$$+ \omega(\omega'_x xy - \omega'_y x^2)\boldsymbol{k}$$

we obtain

$$N_{\text{cor}} = -2\int dm\boldsymbol{r} \times (\omega \times \boldsymbol{v})$$

$$= 2\omega[(\omega'_y I_{xz} - \omega'_z I_{xy})\boldsymbol{i} + (\omega'_z I - \omega'_x I_{xz})\boldsymbol{j} + (\omega'_x I_{xy} - \omega'_y I)\boldsymbol{k}] \tag{7.49}$$

where $I = -\int x^2 dm$ and ω' is the angular velocity of the satellite relative to the xyz axes.

The addition of Eqs. (7.45), (7.47), and (7.49) will give the torque produced on the satellite because of its own motion. However, in these equations, the various I are computed in the moving coordinates. In other words, I changes with time. This is not convenient to apply. It is better to relate I to the principal moments of inertia. Let R be a rotational transformation matrix and I' the principal moment of inertia. Assume that at the beginning of observation, the xyz axes are coincided with the principal axes of the body. Note that

$$\boldsymbol{I}' = \boldsymbol{R}\boldsymbol{I}\boldsymbol{R}^{-1}$$

$$\boldsymbol{I} = \boldsymbol{R}^{-1}\boldsymbol{I}'\boldsymbol{R} \tag{7.50}$$

Now let us consider pitching of the satellite, which means the satellite rotates about the x axis by an angle of θ_p with a speed of $\dot{\theta}_p$. We have

$$\omega'_x = \dot{\theta}_p, \qquad \omega'_y = \omega'_z = 0$$

Because the body is rotated about the x axis counterclockwise by an angle of θ_p, the rotational transformation matrix is

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_p & \sin\theta_p \\ 0 & -\sin\theta_p & \cos\theta_p \end{pmatrix}$$

We find

$$\boldsymbol{I} = \boldsymbol{R}^{-1}\boldsymbol{I}'\boldsymbol{R} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2\cos^2\theta_p + I_3\sin^2\theta_p & (I_2 - I_3)\cos\theta_p\sin\theta_p \\ 0 & (I_2 - I_3)\cos\theta_p\sin\theta_p & I_2\sin^2\theta_p + I_3\cos^2\theta_p \end{pmatrix}$$

Hence

$$I_{xx} = I_1, \qquad I_{xy} = I_{yx} = I_{xz} = I_{zx} = 0$$

$$I_{yz} = (I_2 - I_3)\cos\theta_p \sin\theta_p = I_{zy}$$

Thus the torque produced on the satellite because of pitching is simply

$$\begin{aligned} \boldsymbol{N}_p = \boldsymbol{N}_g &= 3\omega_2(-I_{yz}\boldsymbol{i}) \\ &= -3\omega^2(I_2 - I_3)\cos\theta_p \sin\theta_p \boldsymbol{i} \\ &= -\tfrac{3}{2}\omega^2(I_2 - I_3)\sin 2\theta_p \boldsymbol{i} \end{aligned} \tag{7.51}$$

Next, let us consider rolling of the satellite about the y axis by an angle of θ_R with a speed of $\dot{\theta}_R$, i.e.,

$$\omega'_x = 0, \qquad \omega'_y = \dot{\theta}_R, \qquad \omega'_z = 0$$

then we have

$$\boldsymbol{R} = \begin{pmatrix} \cos\theta_R & 0 & -\sin\theta_R \\ 0 & 1 & 0 \\ \sin\theta_R & 0 & \cos\theta_R \end{pmatrix}$$

$$\boldsymbol{I} = \boldsymbol{R}^{-1}\boldsymbol{I}'\boldsymbol{R} = \begin{pmatrix} I_1\cos^2\theta_R + I_3\sin^2\theta_R & 0 & (-I_1 + I_3)\cos\theta_R \sin\theta_R \\ 0 & I_2 & 0 \\ (-I_1 + I_3)\cos\theta_R \sin\theta_R & 0 & I_1\sin^2\theta_R + I_3\cos^2\theta_R \end{pmatrix}$$

or

$$I_{xx} = I_1\cos^2\theta_R + I_3\sin^2\theta_R$$

$$I_{yy} = I_2$$

$$I_{zz} = I_1\sin^2\theta_R + I_3\cos^2\theta_R$$

$$I_{xz} = I_{zx} = (-I_1 + I_3)\cos\theta_R \sin\theta_R$$

$$I_{xy} = I_{yx} = I_{yz} = I_{zy} = 0$$

The I in Eq. (7.49) is

$$\begin{aligned} I &= -\int x^2 dm = -\frac{1}{2}\int (r^2 + x^2 - y^2 - z^2) dm \\ &= -\frac{1}{2}\int [(r^2 - y^2) - (r^2 - x^2) + (r^2 - z^2)] dm \\ &= -\frac{1}{2}[I_{yy} - I_{xx} + I_{zz}] \\ &= -\frac{1}{2}[I_2 - (I_1\cos^2\theta_R + I_3\sin^2\theta_R) + (I_1\sin^2\theta_R + I_3\cos^2\theta_R)] \\ &= -\frac{1}{2}[I_2 - (I_1 - I_3)\cos 2\theta_R] \end{aligned}$$

With the use of I as just obtained, we find the torque produced on the satellite because of rolling is

$$\begin{aligned}N_{\mathrm{rol}} &= N_g + N_{\mathrm{cer}} + N_{\mathrm{cor}} \\ &= 3\omega^2 I_{xz}\boldsymbol{j} + \omega^2 I_{xz}\boldsymbol{j} + 2\omega(\dot{\theta}_R I_{xz}\boldsymbol{i} - \dot{\theta}_R I\boldsymbol{k}) \\ &= -\omega\dot{\theta}_R(I_1 - I_3)\sin 2\theta_R\,\boldsymbol{i} - 2\omega^2(I_1 - I_3)\sin 2\theta_R\boldsymbol{j} \\ &+ \omega\dot{\theta}_R[I_2 - (I_1 - I_3)\cos(2\theta_R)]\boldsymbol{k} \end{aligned} \tag{7.52}$$

Similarly we can find that the torque acting on the satellite because of yawing about the z axis is

$$\begin{aligned}N_{\mathrm{yaw}} &= -\omega\dot{\theta}_y(I_1 - I_2)\sin(2\theta_y)\boldsymbol{i} - \omega\dot{\theta}_y[I_3 - (I_1 - I_2)\cos(2\theta_y)]j \\ &-\tfrac{1}{2}\omega^2(I_1 - I_2)\sin(2\theta_y)\boldsymbol{k}\end{aligned} \tag{7.53}$$

Therefore, rolling and yawing can produce rotations about all three axes.

From here one can easily suggest a project that is to carry out the proper operational procedure so that the torques generated by the motions of the satellite are balanced. Furthermore, it is easy to recognize the need for a great deal more research for a satellite in an elliptical orbit.

Problems

7.1. Prove that

$$(\dot{\boldsymbol{n}}\boldsymbol{n} - \boldsymbol{n}\dot{\boldsymbol{n}}) = (\boldsymbol{n} \times \dot{\boldsymbol{n}}) \times \overleftrightarrow{1}$$

7.2. Verify Eq. (7.17) through direct evaluation in detail of $(\mathrm{d}\overleftrightarrow{R}/\mathrm{d}t) \cdot \overleftrightarrow{R}^T$.

7.3. Given

$$\omega = \dot{\beta}\boldsymbol{n} + (1 - \cos\beta)(\boldsymbol{n} \times \dot{\boldsymbol{n}}) + \sin\beta\dot{\boldsymbol{n}}$$

prove that by assuming $\boldsymbol{n} \cdot \dot{\boldsymbol{n}} = 0$

$$\boldsymbol{n} \times \omega = -(1 - \cos\beta)\dot{\boldsymbol{n}} + \sin\beta(\boldsymbol{n} \times \dot{\boldsymbol{n}})$$

and

$$\boldsymbol{n} \times (\boldsymbol{n} \times \omega) = -\sin\beta\dot{\boldsymbol{n}} - 2\sin^2(\beta/2)(\boldsymbol{n} \times \dot{\boldsymbol{n}})$$

Consequently,

$$\dot{\boldsymbol{n}} = -\frac{1}{2}\left\{(\boldsymbol{n} \times \omega) + \cot\frac{\beta}{2}[(\boldsymbol{n} \cdot \omega)\boldsymbol{n} - \omega]\right\}$$

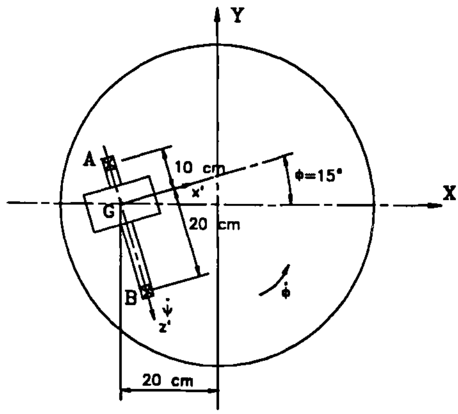

Fig. P7.4

7.4. A round plate rotates about the z axis perpendicular to the x–y plane with an angular velocity $\dot{\phi}$. Mounted on this revolving plate are two bearings A and B that retain a shaft and mass rotating at the angular velocity $\dot{\psi}$ as shown in Fig. P7.4. An $x'y'z'$ system is selected and fixed to the shaft and mass in such a way that the z' axis is along the shaft, x' is perpendicular to the z' axis, and y' is parallel to the Z axis. The mass center G defines the center of this system. The angular velocity $\dot{\psi}$ is observed from a position on the rotating plate. Let the mass be 10 kg, its radius of gyration be $r = 10$ cm, and its angular velocity $\dot{\psi} = 350$ rad/s. Using $\dot{\phi} = 5$ rad/s in the direction shown, find the bearing reactions.

7.5. The rotor of a jet airplane engine is supported by two bearings as shown in Fig. P7.5. The rotor assembly, consisting of the shaft, compressor, and turbine, has a mass of 820 kg and a moment of inertia with respect to its shaft of 45 kg-m^2; its center of mass is lying at point G. The rotor is rotating at 10,000 rpm cw when viewed from the rear. The speed of the airplane is 970 km/h, and it is pulling out of a dive along a path 1530 m in radius. Determine the magnitude and direction of

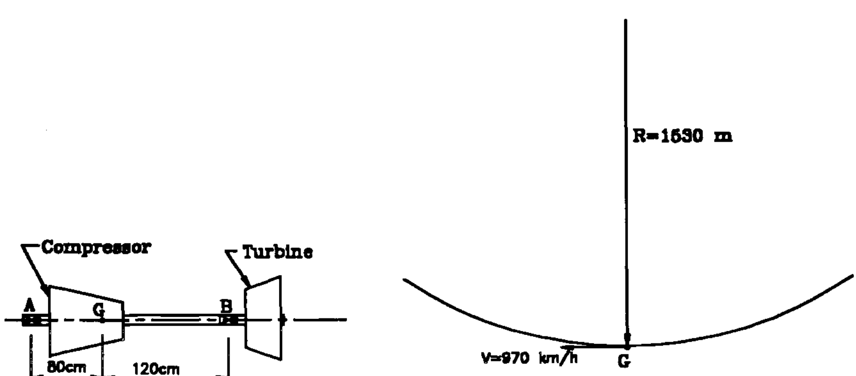

Fig. P7.5

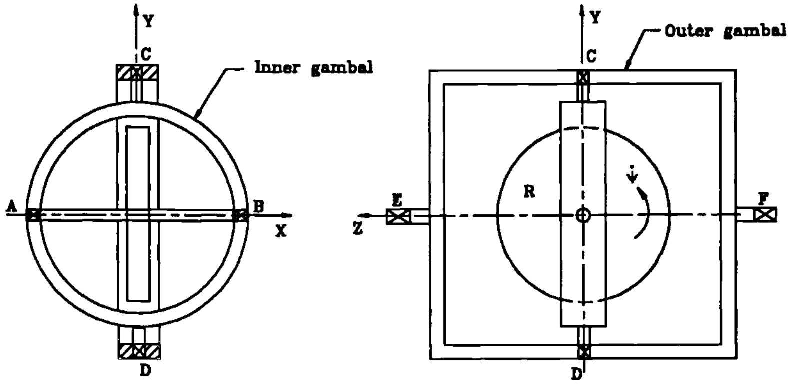

Fig. P7.7

the combined forces that the shaft exerts against the bearings due to the gyroscopic effect and the centrifugal effect.

7.6. The jet airplane in Problem 7.5 is traveling at 850 km/h in a horizontal plane and makes a clockwise turn of 2.0 km radius when viewed from above. The rotor is rotating at 9000 rpm cw when viewed from the rear. Determine the magnitude and direction of the gyroscopic forces that the shaft exerts against the bearings. Will the forces make the front of the plane tilt upward or downward?

7.7. In Fig. P7.7 a gyroscope used in instrument applications is illustrated. The rotor R is mounted in gimbals so that it is free to rotate about all three axes. In the figure A, B, C, D, E, and F are precision bearings. The rotor has a moment of inertia with respect to its axis $I = 0.0025$ kg-m^2 and is rotating at 12,000 rpm. Suppose that the instrument experiences a precession of 1 deg/h about the Z axis. Determine the magnitude and direction of the torque applied to cause the precession.

7.8. A heavy symmetric top is spun with its axis of symmetry in the vertical position initially. Find the conditions that will cause the top to remain vertical.

7.9. Derive Lagrange's equation for the coordinate θ of a heavy symmetrical top. Then solve this relation for the precession angular velocity $\dot{\phi}$ when there are no nutation velocity and acceleration present. From this result, show that there is a minimum valve of ω_z for which precession is possible. Finally, for ω_z higher than the minimum value, show that there are two permissible values of $\dot{\phi}$, corresponding to the cases of fast and slow precession.

7.10. Show that the total torque in yawing motion of a spacecraft in a circular orbit is given by Eq. (7.53).

7.11. Find the torques produced on a satellite in an elliptical orbit caused by its motions of pitching, rolling, and yawing.

8
Fundamentals of Small Oscillations

VIBRATION can be either destructive or beneficial to our daily life. The fatigue of a material, which may lead to the failure of a structure, is possibly caused by vibration. A machine is intentionally designed to be free of vibrations, but sometimes undesirable vibrations just cannot be avoided when it is in service. When a car is driven on the road, an unbalanced wheel or an out-of-round tire can cause it to vibrate. On the other hand, because of the oscillation of its pendulum, a mechanical clock can tell the time. Because of the vibration of its membrane, a loud speaker can produce music.

Because vibrations can be either useful or troublesome, it is desirable that we understand the causes and phenomena of vibrations and further how to control them according to our wishes. Developing the knowledge to accomplish this control is the purpose of Chapters 8 and 9.

As in previous chapters, the required mathematics for studying vibration is presented at the beginning of the chapter. The subjects of the mathematics needed are Fourier series, Fourier integral, and Fourier and Laplace transforms. They are presented in Sections 8.1 and 8.2. Because more functions can satisfy the conditions for the Laplace transform than for the Fourier transform, the Laplace transform method can be applied to many more cases. Section 8.3 presents some important properties of the Laplace transform. Tables of Laplace and Fourier transforms are included in Appendix F. However, we will not deal with the inverse Laplace transform in this chapter because the derivation of formulas involves some lengthy details from the theory of complex variables. A brief description of the inverse transform for some functions is given in Appendix G. The applications of Fourier and Laplace transforms are presented in this chapter. In Chapter 9, we will present applications of Fourier series and more applications of the Laplace transform.

In Section 8.4, we shall study forced vibration systems with single degrees of freedom.These systems are either with damping or without damping and are either harmonically or arbitrarily excited. Because a periodic force can be expanded into a Fourier series, an analysis for one harmonic excitation will suffice to demonstrate that for any other harmonic excitations. Applications of these vibration systems are presented as examples that include accelerometer, seismometer, and packaging. The meaning of the Richter scale, which is a measure of the magnitude of an earthquake, is explained in Example 8.6.

Transient vibration is studied in Section 8.5. This type of vibration is caused by a nonperiodic force. Depending on the type of forcing function applied, response of a general excitation may not be obtained by analytical integration; it can be always integrated numerically through the formulation of arbitrary excitations. The responses of the cases studied in this section are obtained by analytical integration.

Response and velocity spectra of transient vibration are studied in Section 8.6. Because design of a vibration system is often restricted by the maximum response, response spectra may be used for modifying the design so that the maximum response is within the acceptable range.

Section 8.7 is specifically devoted to the application of the Fourier transform for analyzing the response of a vibration system. As we shall study later, the amplitude of vibration as a function of time will be converted into that as a function of frequency by using Fourier transform. Because of limited time in practice, the method used in the vibration analyzer is modified and is called the discrete Fourier transform. Through this, many random vibrations can be analyzed and the amplitude of vibration can be displayed as a function of frequency. From there, we can detect the source of vibration.

8.1 Fourier Series and Fourier Integral

Fourier Series

A Fourier series is a useful tool for solving differential equations and for treating various problems involving periodic functions. It is an infinite series of trigonometric functions and, in general, is expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \tag{8.1}$$

where n is an integer, x can be any value from $-\infty$ to ∞, and a_n and b_n are coefficients.

A function that can be expanded into a Fourier series must satisfy the following conditions: 1) The function is periodic or 2) The function is piecewise continuous between x and $x + 2L$.

A function $f(x)$ is said to be periodic if it is defined for all x with a period of $2L$ such that

$$f(x + 2L) = f(x)$$

The function $f(x)$ shown in Fig. 8.1 is a piecewise continuous function. Note that it is impossible to expand a discrete function as shown in Fig. 8.2 into a Fourier series.

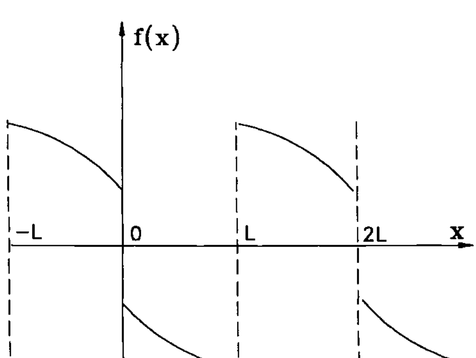

Fig. 8.1 Periodic and piecewise continuous function.

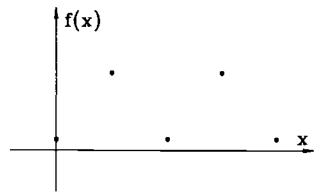

Fig. 8.2 Periodic and discrete function.

To facilitate the determination of the coefficients in a Fourier series, we introduce the following formulas for integrating trigonometric functions:

$$\begin{aligned}
&\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,\mathrm{d}x \\
&= \int_{-L}^{L}\left[\frac{1}{2}\cos(m+n)\frac{\pi x}{L} + \frac{1}{2}\cos(m-n)\frac{\pi x}{L}\right]\mathrm{d}x \\
&= \frac{1}{2}\frac{L}{(m+n)\pi}\sin(m+n)\frac{\pi x}{L}\bigg|_{-L}^{L} + \frac{1}{2}\frac{L}{(m-n)\pi}\sin(m-n)\frac{\pi x}{L}\bigg|_{-L}^{L} \\
&= 0 \qquad \text{if } m \neq n \qquad (m, n \text{ are integers}) \qquad (8.2)
\end{aligned}$$

If $m = n$, then

$$\begin{aligned}
\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,\mathrm{d}x &= \int_{-L}^{L}\cos^2\frac{m\pi x}{L}\,\mathrm{d}x \\
&= \int_{-L}^{L}\frac{1}{2}\left(1+\cos 2m\frac{\pi x}{L}\right)\mathrm{d}x = L \qquad (8.3)
\end{aligned}$$

$$\begin{aligned}
&\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x \\
&= \int_{-L}^{L}\left[\frac{1}{2}\cos(m-n)\frac{\pi x}{L} - \frac{1}{2}\cos(m+n)\frac{\pi x}{L}\right]\mathrm{d}x \\
&= \frac{L}{2(m+n)\pi}\sin(m-n)\frac{\pi x}{L}\bigg|_{-L}^{L} - \frac{L}{2(m-n)\pi}\sin(m+n)\frac{\pi x}{L}\bigg|_{-L}^{L} \\
&= 0 \qquad \text{if } m \neq n \qquad (8.4)
\end{aligned}$$

If $m = n$, then

$$\begin{aligned}
\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x &= \int_{-L}^{L}\sin^2\frac{m\pi x}{L}\,\mathrm{d}x \\
&= \int_{-L}^{L}\frac{1}{2}\left(1-\cos\frac{2m\pi x}{L}\right)\mathrm{d}x = L \qquad (8.5)
\end{aligned}$$

$$\begin{aligned}
&\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx \\
&= \int_{-L}^{L}\left[\frac{1}{2}\sin(m+n)\frac{\pi x}{L} + \frac{1}{2}\sin(m-n)\frac{\pi x}{L}\right]dx \\
&= -\frac{1}{2}\left[\frac{L}{(m+n)\pi}\cos(m+n)\frac{\pi x}{L} + \frac{L}{(m-n)\pi}\cos(m-n)\frac{\pi x}{L}\right]_{-L}^{L} \\
&= 0 \qquad \text{if } m \neq n
\end{aligned} \tag{8.6}$$

If $m = n$, then

$$\begin{aligned}
\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx &= \int_{-L}^{L}\frac{1}{2}\sin\frac{2m\pi x}{L}\,dx \\
&= -\frac{L}{4m\pi}\left[\cos\frac{2m\pi x}{L}\right]_{-L}^{L} \\
&= 0
\end{aligned} \tag{8.7}$$

If $m = n = 0$,

$$\int_{-L}^{L} \cos o.x \cos o.x\,dx = \int_{-L}^{L} dx = 2L \tag{8.8}$$

$$\int_{-L}^{L} \sin Ox \sin Ox\,dx = 0 \tag{8.9}$$

$$\int_{-L}^{L} \sin Ox \cos Ox\,dx = 0 \tag{8.10}$$

Now we can conclude:

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = L\delta_{m,n} \tag{8.11}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = L\delta_{m,n} \tag{8.12}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0 \tag{8.13}$$

Equations (8.11–8.13) are known as orthogonality conditions.

Calculation of the coefficients in a Fourier series. If a function $f(x)$ satisfies the conditions for the Fourier series, it can be expanded into the form of Eq. (8.1). To determine the coefficient a_o, we multiply both sides of Eq. (8.1) by

$\cos 0x = 1$ and integrate from $-L$ to L:

$$\int_{-L}^{L} f(x)\,\mathrm{d}x = \int_{-L}^{L} \frac{1}{2} a_o \,\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\frac{n\pi x}{L} \cos 0x \,\mathrm{d}x$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\frac{n\pi x}{L} \cos 0x \,\mathrm{d}x$$

By using the orthogonality conditions, we get

$$\int_{-L}^{L} f(x)\,\mathrm{d}x = a_o L$$

or

$$a_o = \frac{1}{L} \int_{-L}^{L} f(x)\,\mathrm{d}x \tag{8.14}$$

To determine the coefficient a_n, we multiply both sides of Eq. (8.1) by $\cos(m\pi x/L)$ and integrate from $-L$ to L:

$$\int_{-L}^{L} f(x) \cos\frac{m\pi x}{L} \mathrm{d}x = \frac{1}{2} a_o \int_{-L}^{L} \cos\frac{m\pi x}{L} \,\mathrm{d}x$$
$$+ \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} \,\mathrm{d}x$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L} \,\mathrm{d}x$$

Because

$$\int_{-L}^{L} \cos\frac{m\pi x}{L} \,\mathrm{d}x = 0$$
$$\sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} \,\mathrm{d}x = L a_m$$
$$\sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L} = 0$$

we obtain

$$\int_{-L}^{L} f(x) \cos\frac{m\pi x}{L} \,\mathrm{d}x = L a_m$$

The index m is a dummy index that can be replaced by any symbol. It is convenient to use n, so that

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad n = 0, 1, 2, \ldots \tag{8.15}$$

Note that Eq. (8.15) includes the expression of Eq. (8.14). Similarly, to determine the coefficient b_n, we multiply both sides of Eq. (8.1) by $\sin(m\pi x/L)$ and integrate from $-L$ to L. We find

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \quad n = 1, 2, 3, \ldots \tag{8.16}$$

Therefore, if $f(x)$ is a periodic function and is piecewise continuous, then it can be expanded into a Fourier series as

$$f(x) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \quad n = 1, 2, 3, \ldots$$

It is worthwhile to mention that the integral limits in the preceding equations are not necessarily $-L$ and L. Because $f(x)$ is a periodic function of period of $2L$, the integral limits can be replaced by α and $\alpha + 2L$ where α is any real constant.

Example 8.1

Consider a function $f(x)$ that is known as a square wave and is defined as

$$f(x) = -h \qquad -L < x < o$$

$$f(x) = h \qquad o < x < L$$

where h is a constant. The function is periodic and is shown in Fig. 8.3. The

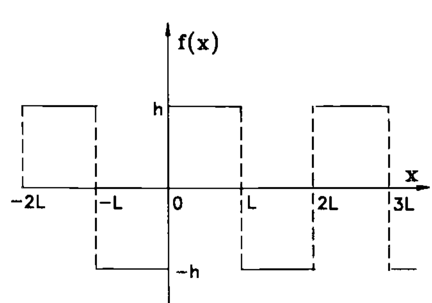

Fig. 8.3 Plot of the function.

interval $-L < x < L$ is so called the Fourier interval. Expand the function into a Fourier series.

Solution. Because this function is periodic and piecewise continuous, we can expand it into a Fourier series in the form of Eq. (8.1). The coefficients are determined as follows:

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x \\
&= \frac{1}{L}\int_{-L}^{0}(-h)\cos\frac{n\pi x}{L}\,\mathrm{d}x + \frac{1}{L}\int_{0}^{L} h\cos\frac{n\pi x}{L}\,\mathrm{d}x \\
&= -\frac{h}{n\pi}\left(\sin\frac{n\pi x}{L}\right)\Big|_{-L}^{0} + \frac{h}{n\pi}\left(\sin\frac{n\pi x}{L}\right)\Big|_{0}^{L} = 0
\end{aligned}
$$

that is true for all the values of n from 1 on up. For $n = 0$, we have zero divided by zero, hence we determine a_0 separately and we find

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\mathrm{d}x = \frac{1}{L}\int_{-L}^{0}(-h)\mathrm{d}x + \frac{1}{L}\int_{0}^{L} h\,\mathrm{d}x \\
&= \frac{1}{L}(-hl + hL) = 0
\end{aligned}
$$

To determine the coefficients b_n, we use Eq. (8.16)

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x \\
&= \frac{1}{L}\left[\int_{-L}^{0}(-h)\sin\frac{n\pi x}{L}\,\mathrm{d}x + \int_{0}^{L} h\sin\frac{n\pi x}{L}\,\mathrm{d}x\right] \\
&= \frac{h}{n\pi}[1-\cos(n\pi)] - \frac{h}{n\pi}[\cos(n\pi)-1] \\
&= \frac{2h}{n\pi}[1-\cos(n\pi)] \\
&= \frac{2h}{n\pi}[1-(-1)^n]
\end{aligned}
$$

Hence

$$
\begin{aligned}
b_n &= 4h/n\pi \qquad \text{as } n = 1, 3, 5 \ldots \\
b_n &= 0 \qquad \text{as } n = 2, 4, 6, \ldots
\end{aligned}
$$

Therefore, the Fourier series for the square wave can be expressed as

$$
f(x) = \frac{4h}{\pi}\left[\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \cdots\right]
$$

or

$$f(x) = \frac{4h}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin\frac{(2m-1)\pi x}{L}$$

Fourier sine series and cosine series. If a function $f(x)$ satisfies the conditions for the Fourier series and is an odd function, i.e.,

$$f(-x) = -f(x)$$

then it can be expanded into a Fourier sine series as

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \tag{8.17}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x \quad n = 1, 2, 3, \ldots \tag{8.18}$$

On the other hand, if a function $f(x)$ satisfies the condition for Fourier series and is an even function, $f(-x) = f(x)$, then it can be expanded into a Fourier cosine series as

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \tag{8.19}$$

where

$$A_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x \tag{8.20}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x \quad n = 1, 2, 3, \ldots \tag{8.21}$$

Fourier Integral

We have studied that a periodic and piecewise continuous function in a finite interval can be represented by a Fourier series. Now we shall generalize the method of Fourier series to include a piecewise continuous function as defined in an infinite interval. If a function $f(x)$ is piecewise continuous defined in the interval $0 < x < \infty$, and is an odd function, then we can write

$$f_0(x) = \sum_{n=0}^{\infty} B_n \frac{L}{\pi}\left(\sin\frac{n\pi x}{L}\right)\frac{\pi}{L} \tag{8.22}$$

For the ease of mathematical operation, we make changes in symbols in the preceding equation and let

$$B_n\frac{L}{\pi} = B(u_n) \qquad \frac{n\pi}{L} = u_n$$

and

$$\Delta u_n = u_{n+1} - u_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} = \Delta u$$

Then we have

$$f_0(x) = \sum_{n=0}^{\infty} B(u_n)(\sin u_n x)\Delta u \tag{8.23}$$

Now, considering the case as L approaches ∞, we have

$$\Delta u = \mathrm{d}u$$

and write u_n as u. Equation (8.23) becomes

$$f_0(x) = \int_0^{\infty} B(u)\sin ux\,\mathrm{d}u \tag{8.24}$$

where

$$B(u) = \lim_{L\to\infty} \frac{L}{\pi} B_n = \lim_{L\to\infty} \frac{L}{\pi}\left[\frac{2}{L}\int_0^L f_0(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x\right]$$
$$= \lim_{L\to\infty} \frac{2}{\pi}\int_0^L f_0(x)\sin ux\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\infty} f_0(x)\sin ux\,\mathrm{d}x \tag{8.25}$$

In the preceding equation, x is a dummy variable that can be replaced by any symbol. By changing x to t, and combining Eqs. (8.24) and (8.25), we obtain

$$f_0(x) = \frac{2}{\pi}\int_0^{\infty} \sin ux\left[\int_0^{\infty} f_0(t)\sin ut\,\mathrm{d}t\right]\mathrm{d}u \tag{8.26}$$

This is known as the Fourier sine integral representation of $f_0(x)$.

Similarly, if $f(x)$ is piecewise continuous defined in the interval $0 < x < \infty$ and is an even function, then it can be represented by

$$f_e(x) = \int_0^{\infty} A(u)\cos ux\,\mathrm{d}u \tag{8.27}$$

where

$$A(u) = \frac{2}{\pi}\int_0^{\infty} f_e(t)\cos ut\,\mathrm{d}t \tag{8.28}$$

or

$$f_e(x) = \frac{2}{\pi}\int_0^{\infty} \cos ux \int_0^{\infty} f_e(t)\cos ut\,\mathrm{d}t\,\mathrm{d}u \tag{8.29}$$

In general, a function always can be expressed as a combination of even and odd functions, i.e.,

$$f(x) = f_e(x) + f_0(x)$$

Using Eqs. (8.26) and (8.29) for even and odd functions, we have

$$\begin{aligned}
f(x) &= \frac{2}{\pi}\left[\int_0^\infty \cos ux \int_0^\infty f_e(t)\cos ut\,\mathrm{d}t\,\mathrm{d}u \right.\\
&\quad \left. + \int_0^\infty \sin ux \int_0^\infty f_0(t)\sin ut\,\mathrm{d}t\,\mathrm{d}u\right]\\
&= \frac{2}{\pi}\left\{\int_0^\infty \cos ux \frac{1}{2}\int_{-\infty}^\infty [f(t) - f_0(t)]\cos ut\,\mathrm{d}t\,\mathrm{d}u \right.\\
&\quad \left. + \int_0^\infty \sin ux \frac{1}{2}\int_{-\infty}^\infty [f(t) - f_e(t)]\sin ut\,\mathrm{d}t\,\mathrm{d}u\right\}\\
&= \frac{1}{\pi}\left\{\int_0^\infty \cos ux \int_{-\infty}^\infty f(t)\cos ut\,\mathrm{d}t\,\mathrm{d}u \right.\\
&\quad \left. + \int_0^\infty \sin ux \int_{\infty}^\infty f(t)\sin ut\,\mathrm{d}t\,\mathrm{d}u\right\}\\
&= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)[\cos ux \cos ut + \sin ux \sin ut]\,\mathrm{d}t\,\mathrm{d}u\\
&= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos u(x - t)\,\mathrm{d}t\,\mathrm{d}u \qquad (8.30)
\end{aligned}$$

Because the integrand is an even function of u, we can rewrite Eq. (8.30) in the form of

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\cos u(x - t)\,\mathrm{d}t\,\mathrm{d}u \qquad -\infty < x < \infty \qquad (8.31)$$

This expression is known as the complete Fourier integral representation of $f(x)$ for all values of x. The conditions for a function to be expressed in Eq. (8.31) are 1) the integral $\int_{-\infty}^{\infty}|f(t)|\mathrm{d}t$ must exist, and 2) $f(t)$ must be a single-valued function of the real variable t throughout the range $-\infty < t < \infty$. It may have several finite discontinuities.

Example 8.2

1) Find the Fourier sine integral representation of the function $f(x)$ which is given as

$$f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & a < x < \infty \end{cases}$$

$$f(-x) = -f(x)$$

and is shown in Fig. 8.4.

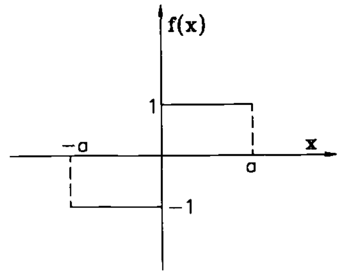

Fig. 8.4 Plot of the function.

2) Evaluate each term in the Fourier sine integral representation and prove that the result of the expression truly represents the original function as shown in Fig. 8.4.

Solution. 1) From Eq. (8.26), we have

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty \sin ux \left[\int_0^\infty f(t)\sin ut\,\mathrm{d}t\right]\mathrm{d}u \\ &= \frac{2}{\pi}\int_0^\infty \sin ux \int_0^a \sin ut\,\mathrm{d}t\,\mathrm{d}u \\ &= \frac{2}{\pi}\int_0^\infty \sin ux \left[-\frac{\cos ut}{u}\right]_0^a \mathrm{d}u \\ &= \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos ua}{u}\right)\sin ux\,\mathrm{d}u \end{aligned}$$

The result in the preceding equation is the one that we are looking for.

2) Rewrite the preceding equation as

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin ux}{u} - \frac{\cos ua \sin ux}{u}\right)\mathrm{d}u = \frac{2}{\pi}(I_1 - I_2)$$

where

$$I_1 = \int_0^\infty \frac{\sin ux}{u}\,\mathrm{d}u$$

$$I_2 = \int_0^\infty \frac{\cos ua \sin ux}{u}\,\mathrm{d}u$$

Evaluate I_1, we find

$$I_1 = \int_0^\infty \frac{\sin ux}{ux}\,\mathrm{d}(ux) = \int_0^\infty \frac{\sin z}{z}\mathrm{d}z = \frac{\pi}{2} \qquad \text{as } x > 0$$

As $x < 0$,

$$I_1 = \int_0^{-\infty} \frac{\sin z}{z}\,\mathrm{d}z$$

$$= -\int_0^{\infty} \frac{\sin z^*}{z^*}\,\mathrm{d}z \qquad (\text{when } z = -z^*)$$

$$= -\frac{\pi}{2}$$

On the other hand, we rewrite I_2 in two parts as

$$I_2 = \frac{1}{2}\int_0^{\infty} \frac{\sin(x+a)u + \sin(x-a)u}{u}\,\mathrm{d}u$$

$$= I_{21} + I_{22}$$

Evaluate I_{21}, we obtain

$$I_{21} = \frac{1}{2}\int_0^{\infty} \frac{\sin(x+a)u}{u}\,\mathrm{d}u$$

$$= \frac{1}{2}\int_0^{\infty} \frac{\sin(x+a)u}{(x+a)u}\,\mathrm{d}[(x+a)u] = \frac{\pi}{4} \qquad \text{as } x > -a$$

$$I_{21} = -\frac{\pi}{4} \qquad \text{as } x < -a$$

Similarly, we have

$$I_{22} = \frac{1}{2}\int_0^{\infty} \frac{\sin(x-a)u}{u}\,\mathrm{d}u$$

$$= \frac{1}{2}\int_0^{\infty} \frac{\sin(x-a)u}{(x-a)u}\,\mathrm{d}[(x-a)u] = \frac{\pi}{4} \qquad \text{as } x > a$$

$$I_{22} = -\frac{\pi}{4} \qquad \text{as } x < a$$

Now the function $f(x)$ can be expressed as

$$f(x) = \frac{2}{\pi}[I_1 - (I_{21} + I_{22})]$$

We find that

$$f(x) = \frac{2}{\pi}\left[-\frac{\pi}{2} - \left(-\frac{\pi}{4} - \frac{\pi}{4}\right)\right] = 0$$

as $x < -a$,

$$f(x) = \frac{2}{\pi}\left[-\frac{\pi}{2} - \left(-\frac{\pi}{4} - \frac{\pi}{4}\right)\right] = -1$$

as $-a < x < 0$,

$$f(x) = \frac{2}{\pi}\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{\pi}{4}\right)\right] = 1$$

as $0 < x < a$, and

$$f(x) = \frac{2}{\pi}\left[\frac{\pi}{2} - \left(\frac{\pi}{4} + \frac{\pi}{4}\right)\right] = 0$$

as $x > a$. The preceding result proves that the Fourier sine integral representation is truly the original function.

Example 8.3

1) Find the Fourier cosine integral representation of the function $f(x)$ that is defined by

$$f(x) = \begin{cases} 1 & \text{as } 0 < x < a \\ 0 & \text{as } a < x < \infty \end{cases}$$

and

$$f(-x) = f(x)$$

2) Evaluate the Fourier cosine integral representation obtained in part 1 and prove that the result is the original function.

Solution. 1) From Eq. (8.29) we have

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty \cos ux \left[\int_0^\infty f(t)\cos ut\,\mathrm{d}t\right]\mathrm{d}u \\ &= \frac{2}{\pi}\int_0^\infty \cos ux \int_0^a \cos ut\,\mathrm{d}t\mathrm{d}u \\ &= \frac{2}{\pi}\int_0^\infty \cos ux \frac{\sin ut}{u}\bigg|_0^a \mathrm{d}u \\ &= \frac{2}{\pi}\int_0^\infty \frac{\sin ua \cos ux}{u}\mathrm{d}u \end{aligned}$$

This is the Fourier cosine integral representation of the given functions.

2) Evaluate the integral, and we find

$$\begin{aligned} I &= \int_0^\infty \frac{\sin ua \cos ux}{u}\mathrm{d}u \\ &= \frac{1}{2}\int_0^\infty \frac{\sin(x+a)u - \sin(x-a)u}{u}\mathrm{d}u = I_1 - I_2 \end{aligned}$$

where

$$I_1 = \frac{1}{2}\int_0^\infty \frac{\sin(x+a)u}{u}\,du$$

$$I_2 = \frac{1}{2}\int_0^\infty \frac{\sin(x-a)u}{u}\,du$$

Note that the integrals have been evaluated in the previous example. The results are collected as follows:

$$I_1 = \begin{cases} \pi/4 & \text{as } x > -a \\ -\pi/4 & \text{as } x < -a \end{cases}$$

$$I_2 = \begin{cases} \pi/4 & \text{as } x > a \\ -\pi/4 & \text{as } x < a \end{cases}$$

Now, the function $f(x)$ can be evaluated by

$$f(x) = \frac{2}{\pi}(I_1 - I_2)$$

We find that

$$f(x) = \frac{2}{\pi}\left[-\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 0$$

as $x < -a$,

$$f(x) = \frac{2}{\pi}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 1$$

as $-a < x < a$, and

$$f(x) = \frac{2}{\pi}\left[\frac{\pi}{4} - \left(\frac{\pi}{4}\right)\right] = 0$$

as $x > a$. The result reached agrees exactly with the original function.

Example 8.4

Find the complete Fourier integral representation of the functions which is given as

$$f(x) = \begin{cases} 0 & \text{as } -\infty < x < 0 \\ 1 & \text{as } 0 < x < a \\ 0 & \text{as } a < x < \infty \end{cases}$$

Solution. From Eq. (8.31), we have the complete Fourier integral representation of $f(x)$ as

$$f(x) = \frac{1}{2\pi}\int_{\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(x-t)\,\mathrm{d}t\,\mathrm{d}u$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{a} [\cos ux \sin ut + \sin ux \sin ut]\,\mathrm{d}t\,\mathrm{d}u$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\cos ux\frac{\sin ut}{u} - \sin ux\frac{\cos ut}{u}\right]_0^a \mathrm{d}u$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\cos ux \sin ua}{u} + \sin ux\frac{1-\cos ua}{u}\right]\mathrm{d}u$$

Note that the integrand is an even function of u. Hence we can write

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\frac{\cos ux \sin ua}{u} + \frac{\sin ux(1-\cos ua)}{u}\right]\mathrm{d}u$$

This result is the combination of the Fourier cosine and sine integral representations found in Examples 8.3 and 8.2 except the coefficient is reduced to one half, because the given function in this example is one half of the sum of the functions in the previous examples. In addition, it is worthwhile to point out that the expression that was reached also represents the inverse Fourier transform of the transformed function. Details of the Fourier transform are discussed in Section 8.2.

8.2 Fourier and Laplace Transforms

The Fourier transform is a powerful tool for solving differential equations. When it is applied to linear ordinary differential equations with constant coefficients, the differential equation is converted into an algebraic equation. Then the solution of the original differential equation is then reduced to the calculation of inverse transforms of the transformed functions obtained from the algebraic equation.

The Fourier transform also can be applied to linear partial differential equations with constant coefficients. If it is applied to one particular independent variable, the number of independent variables in the partial differential equations is reduced by one. The coefficients in the differential equations are not restricted to constants. Constant coefficients are simpler to apply to the method. The Laplace transform may be considered as a special case of the Fourier transform. Both transforms are closely related to the Fourier series and integral discussed in the previous section. To see how they are related, we shall begin the discussion with the complete Fourier integral representation of a function.

By Eq. (8.31), the complete Fourier integral representation of $f_1(x)$ is

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t)\cos u(x-t)\,\mathrm{d}t\,\mathrm{d}u$$

Because

$$\cos u(x-t) = \tfrac{1}{2}\left[e^{iu(x-t)} + e^{-iu(x-t)}\right]$$

the equation becomes

$$f_1(x) = \frac{1}{2\pi}\left\{\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t)e^{iu(x-t)}\mathrm{d}t\,\mathrm{d}u + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t)e^{-iu(x-t)}\mathrm{d}t\,\mathrm{d}u\right\}$$

In the second part of the preceding equation, we make a change in one of the variables. First we change u to $-v$, and then change v back to u, because v is a dummy variable. Finally we have

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t)e^{iu(x-t)}\mathrm{d}t\,\mathrm{d}u$$

or

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}\int_{-\infty}^{\infty} e^{-iut} f_1(t)\,\mathrm{d}t\,\mathrm{d}u$$

Now let the Fourier transform of $f_1(t)$ be defined as

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} e^{-iut} f_1(t)\,\mathrm{d}t \tag{8.32}$$

Then the inverse transform of $\mathcal{F}(u)$ is

$$f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iut}\mathcal{F}(u)\,\mathrm{d}u \tag{8.33}$$

The conditions for the existence of the Fourier transform of $f_1(t)$ are the same as for the Fourier integral representation stated in Section 8.1.

To show that the Laplace transform is a special case of the Fourier transform, we consider the following function

$$f_1(t) = \begin{cases} 0 & \text{as } -\infty < t < 0 \\ e^{-\gamma t} f(t) & \text{as } 0 < t < \infty \end{cases}$$

where γ is a real number. The Fourier transform of $f_1(t)$ then becomes

$$\mathcal{F}(u) = \int_{0}^{\infty} e^{-(\gamma+iu)t} f(t)\,\mathrm{d}t \tag{8.34}$$

and the inverse transform is

$$\begin{aligned} e^{-\gamma t} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(u)e^{iut}\,\mathrm{d}u \\ f(t) &= \frac{e^{\gamma t}}{2\pi}\int_{-\infty}^{\infty} e^{iut}\mathcal{F}(u)\,\mathrm{d}u \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\gamma+iu)t}\mathcal{F}(u)\,\mathrm{d}u \end{aligned} \tag{8.35}$$

In this equation, we make changes in variables and let $s = \gamma + iu$ and $\mathcal{F}(u) = F(s)$ then $\mathrm{d}s = i\,\mathrm{d}u$. Thus, we write

$$\mathcal{F}(u) = F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

This is defined as the Laplace transform of $f(t)$. The symbol for the transform is

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \tag{8.36}$$

From Eq. (8.35), the inverse Laplace transform is

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s)\,\mathrm{d}s \tag{8.37}$$

The conditions for the existence of the Laplace transform of a function $f(t)$ are as follows:

1) The term $f(t)$ is continuous or piecewise continuous in every finite interval $t_1 \leq t < T$, where $t_1 > 0$ and $T > t_1$.

2) The term $t^n|f(t)|$ is bounded near $t = 0$ for some number n when $n < 1$.

3) The term $e^{-s_o t}|f(t)|$ is bounded for large values of t for some positive real number s_o. Therefore, $f(t)$ may be infinite as $t \to 0$ or finite as $t \to \infty$. The Laplace transform of such a function is still possible.

8.3 Properties of Laplace Transforms

In this section, we shall establish, by detailed calculations, the properties of the Laplace transforms of functions that are very important in solving differential equations. With the sample calculations shown in this section, it would be easy for the reader to establish other formulas of the Laplace transforms given in Appendix F.

The differentiation of $f(t)$ is

$$\mathcal{L}\left[\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right] = s\mathcal{L}[f(t)] - f(0) \tag{8.38}$$

Proof: By using the method of integration by parts, we find that

$$\mathcal{L}\left[\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right] = \int_0^\infty e^{-st}\frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{d}t = e^{-st} f(t)|_0^\infty$$

$$+ s\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = sF(s) - f(0+)$$

where 0+ means on the positive side of zero.

For the second derivative,

$$\mathcal{L}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right] = s^2F(s) - sf(0+) - \frac{\mathrm{d}f}{\mathrm{d}t}(0+) \tag{8.39}$$

Proof:

$$\mathcal{L}\left[\frac{d^2 f}{dt^2}\right] = \int_0^\infty e^{-st}\frac{d^2 f}{dt^2}\,dt = e^{-st}\frac{df}{dt}\bigg|_0^\infty$$
$$+ s\int_0^\infty e^{-st}\frac{df}{dt}\,dt = s[sF(s) - f(0+)] - \frac{df}{dt}(0+)$$
$$= s^2F(s) - sf(0+) - \frac{df}{dt}(0+)$$

The integration of $f(t)$ is

$$\mathcal{L}\left[\int_0^t f(u)\,du\right] = \frac{1}{s}F(s)$$

Proof:

$$\mathcal{L}\left[\int_0^t f(u)\,du\right] = \int_0^\infty e^{-st}\left[\int_0^t f(u)\,du\right]dt$$
$$= \left[\frac{e^{-st}}{-s}\int_0^t f(u)\,du\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f(t)\,dt = \frac{1}{s}F(s) \tag{8.40}$$

If the lower limit in the integral is not zero, then

$$\mathcal{L}\left[\int_a^t f(u)\,du\right] = \mathcal{L}\left[\left(\int_0^t - \int_0^a\right) f(u)\,du\right]$$
$$= \frac{1}{s}F(s) - \frac{1}{s}\int_0^a f(u)\,du \tag{8.41}$$

The translation property is

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

Proof:

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{-st}e^{at}f(t)\,dt$$
$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a) \tag{8.42}$$

If

$$f(t) = \begin{cases} 0 & \text{as } t < a (a \geq 0) \\ g(t-a) & \text{as } t \geq a \end{cases}$$

then

$$F(s) = e^{-as}\,G(s) \tag{8.43}$$

where $G(s)$ is the transformed function of $g(t)$.

Proof:

$$\mathcal{L}[f(t)] = \int_a^{\infty} e^{-st} g(t-a)\mathrm{d}t$$

Let $x = t - a$, then

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-s(x+a)} g(x)\mathrm{d}x$$

$$= e^{-sa}\int_0^{\infty} e^{-sx} g(x)\mathrm{d}x = e^{-sa}G(s)$$

Let

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \tag{8.44}$$

Proof:

$$F(s) = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t$$

$$\frac{\mathrm{d}}{\mathrm{d}s}F(s) = \int_0^{\infty}(-t)e^{-st} f(t)\mathrm{d}t = (-1)\mathcal{L}[tf(t)]$$

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}F(s) = \int_0^{\infty}(-t)^2 e^{-st} f(t)\mathrm{d}t = (-1)^2\mathcal{L}[t^2 f(t)]\cdots$$

Therefore

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n}$$

If $\mathcal{L}[f(t)] = F(s)$ and if $[f(t)/t] \leq (M/t^n)$ as $t \to 0+$ with $n < 1$, $M =$ finite, then

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)\mathrm{d}s \tag{8.45}$$

Proof:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}\mathrm{d}t$$

$$\int_s^{\infty} F(s)\mathrm{d}s = \int_s^{\infty}\left[\int_0^{\infty} f(t)e^{-st}\mathrm{d}t\right]\mathrm{d}s$$

Because s and t are independent, the integration order can be changed:

$$\begin{aligned}\text{LHS} &= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\mathrm{d}s\right]\mathrm{d}t\\ &= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty \mathrm{d}t\\ &= \int_0^\infty \frac{f(t)}{t}e^{-st}\mathrm{d}t = \mathcal{L}\left[\frac{f(t)}{t}\right]\end{aligned}$$

The convolution

$$\mathcal{L}\left[\int_0^t f(t-u)g(u)\mathrm{d}u\right] = F(s)G(s) \tag{8.46}$$

or

$$\mathcal{L}[f * g] = F(s)G(s)$$
$$\mathcal{L}[g * f] = F(s)G(s)$$

Proof:

$$\begin{aligned}F(s)G(s) &= \left[\int_0^\infty e^{-sv}f(v)\mathrm{d}v\right]\left[\int_0^\infty e^{-su}g(u)\mathrm{d}u\right]\\ &= \int_0^\infty\int_0^\infty e^{-s(v+u)}f(v)g(u)\mathrm{d}v\,\mathrm{d}u\\ &= \int_0^\infty g(u)\left[\int_0^\infty e^{-s(v+u)}f(v)\mathrm{d}v\right]\mathrm{d}u\end{aligned}$$

Let $v = t - u$, $\mathrm{d}v = \mathrm{d}t$, then

$$\int_0^\infty e^{-s(v+u)}f(v)\mathrm{d}v = \int_u^\infty e^{-st}f(t-u)\mathrm{d}t$$

Therefore

$$F(s)G(s) = \int_0^\infty\left[\int_u^\infty e^{-st}f(t-u)g(u)\mathrm{d}t\right]\mathrm{d}u$$

The integration shown in Fig. 8.5 can be represented by the triangular area bounded by $t = u$ and $u = 0$. By interchanging the order of integration and changing the

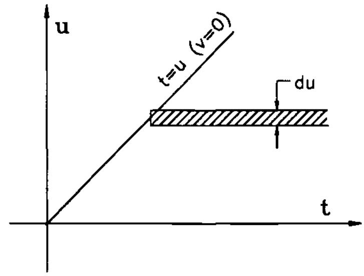

Fig. 8.5 Integration by *t* then *u*.

limits as shown in Fig. 8.6, we can obtain the equivalent result:

$$\begin{aligned}F(s)G(s) &= \int_0^\infty \left[\int_0^t e^{-st} f(t-u)g(u)\,\mathrm{d}u\right]\mathrm{d}t\\ &= \int_0^\infty e^{-st}\left[\int_0^t f(t-u)g(u)\,\mathrm{d}u\right]\mathrm{d}t\\ &= \mathcal{L}\left[\int_0^t f(t-u)g(u)\,\mathrm{d}u\right]\end{aligned}$$

By changing variables in Eq. (8.46) $t - u = v$, we have

$$\begin{aligned}\int_0^t f(t-u)g(u)\,\mathrm{d}u &= \int_t^0 f(v)g(t-v)\,\mathrm{d}(-v)\\ &= \int_0^t f(v)g(t-v)\,\mathrm{d}v\end{aligned}$$

Therefore, we can easily establish

$$f * g = g * f$$

Singularity functions (Dirac delta function) are

$$\mathcal{L}[\delta(t - t_1)] = e^{-t_1 s} \tag{8.47}$$

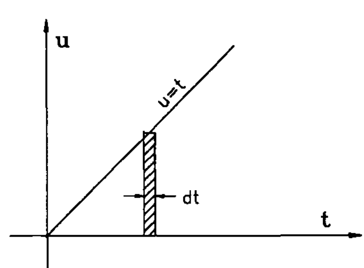

Fig. 8.6 Integration by *u* then *t*.

Proof: The Dirac delta function is defined as

$$\delta(t) = \begin{cases} \lim\limits_{t_0 \to 0} \dfrac{1}{t_0} & \text{as } 0 < t < t_0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\mathcal{L}[\delta(t)] = \int_0^\infty e^{-st}\delta(t)\,\mathrm{d}t = \lim_{t_0 \to 0} \int_0^{t_0} \frac{1}{t_0} e^{-st}\,\mathrm{d}t$$

$$= \lim_{t_0 \to 0} \frac{1}{st_0}(1 - e^{-st_0}) = \lim_{t_0 \to 0} \frac{se^{-st_0}}{s} = 1$$

The Laplace transform of $\delta(t - t_1)$ can be obtained by using Eq. (8.43) and the preceding result. Thus we write

$$\mathcal{L}[\delta(t - t_1)] = e^{-t_1 s}\mathcal{L}[\delta(t)] = e^{-t_1 s}$$

Sometimes, it is useful to know the Laplace transform of the first derivative of Dirac delta function that can be written as

$$\mathcal{L}[\delta'(t - t_1)] = se^{-t_1 s} \tag{8.48}$$

Proof: The first derivative of the Dirac delta function is defined as

$$\delta'(t) = \lim_{t_0 \to 0} \frac{\delta(t) - \delta(t - t_0)}{t_0}$$

which is shown in Fig. 8.7. In other words

$$f(t) = \frac{1}{t_0^2} \qquad \text{as } 0 < t < t_0$$

$$f(t) = -\frac{1}{t_0^2} \qquad \text{as } t_0 < t < 2t_0$$

$$f(t) = 0 \qquad \text{elsewhere}$$

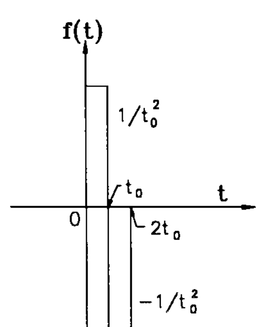

Fig. 8.7 Shape of $\delta'(t)$ before taking limit.

and the derivative of the Dirac delta function is defined as

$$\delta'(t) \equiv \lim_{t_0 \to 0} f(t)$$

Hence, the Laplace transform of the function is

$$\begin{aligned}\mathcal{L}[\delta'(t)] &= \lim_{t_0 \to 0} \frac{1}{t_0^2}\left\{\int_0^{t_0} e^{-st}\,\mathrm{d}t - \int_{t_0}^{2t_0} e^{-st}\,\mathrm{d}t\right\}\\ &= \lim_{t_0 \to 0} \frac{1}{st_0^2}[1 - e^{-st_0} + e^{-2st_0} - e^{-st_0}]\\ &= \lim_{t_0 \to 0} \frac{1}{st_0^2}(1 - e^{-st_0})^2 = s\end{aligned}$$

Again, with the use of Eq. (8.43), we have

$$\mathcal{L}[\delta'(t - t_1)] = e^{-t_1 s}\mathcal{L}[\delta'(t)] = se^{-t_1 s}$$

8.4 Forced Harmonic Vibration Systems with Single Degree of Freedom

Any system consisting of a mass and a spring or the equivalent is capable of vibration. Because of friction, practical problems are usually modeled as a system consisting of a mass, a spring, and a damper as shown in Fig. 8.8. The damper is modeled as a viscous device with damping force proportional to the velocity $\dot{x}$. The weight of the mass is balanced by the spring force $k\Delta$, where k is the spring constant and Δ is the initial deformation of the spring. $F(t)$ is an external force applied to the mass. For the system, the balance of forces gives

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{8.49}$$

To understand the behavior of the system, first we examine the solution of the homogeneous equation:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{8.50}$$

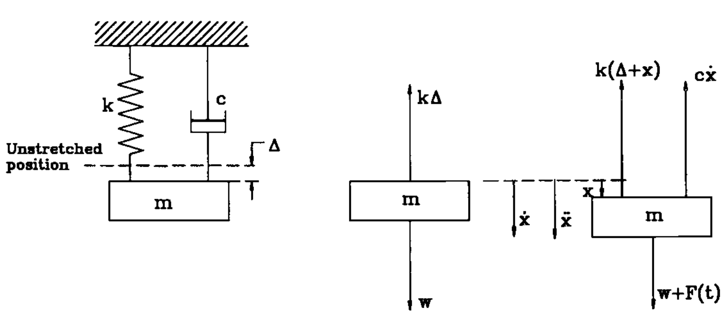

Fig. 8.8 Single-degree-of-freedom system.

Assuming that the solution of the equation is an exponential function of time,

$$x = Xe^{st}$$

where X and s are arbitrary, we find

$$(ms^2 + cs + k)Xe^{st} = 0$$

Because X and e^{st} cannot be zero for all values of t, we must have

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0$$

or

$$S_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

That means

$$\begin{aligned} x &= Ae^{s_1 t} + Be^{s_2 t} \\ &= \exp\left[-\frac{c}{2m}t\right]\left\{A\exp\left[t\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right]\right. \\ &\quad \left. + B\exp\left[-t\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right]\right\} \end{aligned} \tag{8.51}$$

The preceding equation can have three possible results, which are discussed in detail as follows.

Case 1 Overdamped motion

As $(c/2m)^2 > k/m$, the two roots of s are real-negative. The vibration will be damped out rapidly.

Case 2 Critically damped motion

As $(c/2m)^2 = k/m$, the two roots of s are the same. The solution of Eq. (8.50) can be written as

$$x = (C + Dt)e^{-\omega_n t}$$

where ω_n = natural frequency = $\sqrt{k/m}$.

By applying initial conditions we find

$$C = x(0)$$

$$D = \dot{x}(0) + \omega_n x(0)$$

Therefore

$$x = \{x(0) + [\dot{x}(0) + \omega_n x(0)]t\}e^{-\omega_n t} \tag{8.52}$$

The oscillation also will be damped because of $e^{-\omega_n t}$.

Case 3 Underdamped motion

Finally, let us consider the case as $(c/2m)^2 < k/m$. We define the frequency of damped oscillation as

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \tag{8.53}$$

so that

$$\begin{aligned} x &= e^{-\frac{c}{2m}t}[Ae^{i\omega_d t} + Be^{-i\omega_d t}] \\ &= e^{-\frac{c}{2m}t}[c_1 \sin\omega_d t + c_2 \cos\omega_d t] \end{aligned}$$

where

$$c_1 = i(A - B), \qquad c_2 = A + B$$

By applying the initial conditions, the constants are found as

$$c_1 = \frac{1}{\omega_d}\left[\dot{x}(0) + \frac{c}{2m}x(0)\right]$$

and $c_2 = x(0)$. Hence

$$x = \exp\left[-\frac{c}{2m}t\right]\left\{\frac{1}{\omega_d}\left[\dot{x}(0) + \frac{c}{2m}x(0)\right]\sin\omega_d t + x(0)\cos\omega_d t\right\} \tag{8.54}$$

For this case, the amplitude of oscillation is still decreasing continually because of the damping effect, and the frequency of oscillation is reduced from the natural frequency as given in Eq. (8.53).

In all cases, the solution of the homogeneous equation indicates that the amplitude of the vibration will be vanishing as the time increases.

Forced Harmonic Vibration

Harmonic excitation is commonly produced by the unbalance in a rotating machinery. The force is often periodic and is represented by $F(t)$ in Eq. (8.49). Because the periodic function can be expressed in a Fourier series, we can take one term first for $F(t)$ when analyzing the problem. Because the differential equation is linear, the principle of superposition allows us to study a particular case first. The final result is the summation of solutions from all the cases considered.

Now let us consider $F(t) = F_0 \sin\omega t$. From Eq. (8.49), we have

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega t = \mathrm{Im}\,(F_0 e^{i\omega t}) \tag{8.55}$$

where F_0 is the real constant. Let the solution be

$$x = \mathrm{Im}(Ae^{i\omega t})$$

Then we have

$$\dot{x} = \mathrm{Im}(+iA\omega e^{i\omega t})$$
$$\ddot{x} = \mathrm{Im}(-A\omega^2 e^{i\omega t})$$

Substituting the preceding expressions into Eq. (8.55) and dropping the symbol for imaginary, we find

$$(-m\omega^2 + ic\omega + k)Ae^{i\omega t} = F_0 e^{i\omega t}$$
$$A = \frac{F_0/m}{k/m - \omega^2 + (ic\omega/m)}$$
$$= \frac{F_0}{m}\left\{\frac{(k/m - \omega^2) - i(c\omega/m)}{[k/m - \omega^2]^2 + (c\omega/m)^2}\right\}$$

Let

$$D = (k/m - \omega^2)^2 + (c\omega/m)^2$$
$$\cos\phi = \frac{(k/m - \omega^2)}{\sqrt{D}}, \qquad \sin\phi = \frac{c\omega/m}{\sqrt{D}} \tag{8.56}$$

where ϕ is called the phase angle between the vibrating mass and the excitation, then,

$$A = \frac{F_0}{m\sqrt{D}}(\cos\phi - i\sin\phi) = \frac{F_0}{m\sqrt{D}}e^{-i\phi}$$
$$x = \mathrm{Im}\left[\frac{F_0}{m\sqrt{D}}e^{i(\omega t - \phi)}\right]$$
$$= \frac{F_0}{m\sqrt{D}}\sin(\omega t - \phi) \tag{8.57}$$

The general solution is the summation of the homogeneous solution and the particular solution. Therefore, we have

$$x = \frac{F_0}{m\sqrt{D}}\sin(\omega t - \phi) + \exp\left[-\frac{c}{2m}t\right]\left\{\frac{1}{\omega_d}\left[\dot{x}(0) + \frac{c}{2m}x(0)\right]\sin\omega_d t + x(0)\cos\omega_d t\right\} \tag{8.58}$$

If the excitation force is a general periodic function, then

$$f(t) = \sum_i c_i f_i(t) \tag{8.59}$$

where c_i is a known constant and $f_i(t)$ may be any sinusoidal function in the Fourier series, although it is not restricted to sinusoidal functions. Because the principle of superposition is applicable, then, for each force $f_i(t)$, we can write the corresponding displacement as $x_i(t)$ so that the differential equation becomes

$$m\ddot{x}_i + c\dot{x}_i + kx_i = f_i(t) \tag{8.60}$$

After $x_i(t)$ is obtained, the general particular solution can be written as

$$x(t) = \sum_i c_i x_i(t) \tag{8.61}$$

Therefore, the solution given in Eq. (8.57) represents the typical form of the particular solution for the study of a harmonically excited motion. In addition, it is worthwhile to mention that $x(t)$ is the displacement of mass m and is often called the response of the system.

Example 8.5

Consider a spring mass system. The mass M is constrained to move only in the vertical direction. The vibration is excited by a rotating machine with an unbalanced mass m as shown in Fig. 8.9. The unbalanced mass m is at an eccentricity of e and is rotating with angular velocity ω. 1) Formulate an equation for describing the dynamic behavior of the system. 2) Find the amplitude of the vibration of M as a function of the force of excitation $me\omega^2$. 3) Find the expression for the phase angle between the vibration of M and the excitation. 4) Determine also the complete solution of the equation obtained in part 1.

Solution. 1) Let x be the displacement of the mass $(M - m)$ from its static equilibrium position, and $x + e\sin\omega t$ for the displacement of the unbalanced mass m. The equation of motion is

$$(M - m)\ddot{x} + m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x + e\sin\omega t) = -kx - c\dot{x}$$

Rearranging the terms, we have

$$M\ddot{x} + c\dot{x} + kx = (me\omega^2)\sin\omega t \tag{8.62}$$

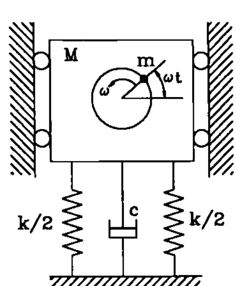

Fig. 8.9 Rotating machine with an unbalanced mass.

2) Note that this equation is identical to Eq. (8.55) except that m is replaced by M and F_0 by $me\omega^2$. Using Eq. (8.57) with the change of symbols, we find

$$x(t) = \frac{me\omega^2}{M\sqrt{D}} \sin(\omega t - \phi)$$

Let X be the amplitude of x, then we write

$$X = \frac{me\omega^2}{M\sqrt{D}} \tag{8.63}$$

where $D = (k/M - \omega^2)^2 + (c\omega/M)^2$.

3) The phase angle ϕ can be determined by Eq. (8.56):

$$\tan\phi = \frac{c\omega/M}{(k/M - \omega^2)} \tag{8.64}$$

Note that the phase angle ϕ is less than 90 deg if $\omega^2 < k/M$ and is greater than 90 deg as $\omega^2 > k/M$.

4) The complete solution is similar to Eq. (8.58) and can be written as

$$\begin{aligned} x(t) &= \frac{me\omega^2}{M\sqrt{D}} \sin(\omega t - \phi) \\ &+ \exp\left(-\frac{ct}{2M}\right)\left\{\frac{1}{\omega_d}\left[\dot{x}(0) + \frac{c}{2M}x(0)\right]\sin\omega_d t + x(0)\cos\omega_d t\right\} \end{aligned} \tag{8.65}$$

Example 8.6

A seismometer is an instrument for measuring the intensity of an earthquake. The design of the instrument is shown in Fig. 8.10. 1) Formulate an equation for describing the dynamic behavior of the instrument due to the ground vibration. 2) Find the relationship between the amplitudes of ground vibration and the instrument. 3) Discuss the essential factors in the design of the seismometer. 4) Suppose that the measured result is 7 on the Richter scale. What is the actual magnitude of the earthquake?

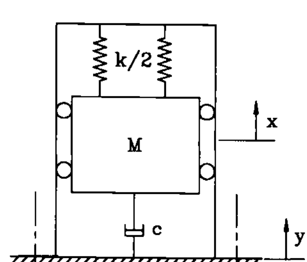

Fig. 8.10 Schematic diagram of a seismometer.

Solution. 1) Assume that the case is rigid and that the displacement of M is x and of the ground is y. The equation of motion for this system can be written as

$$M\ddot{x} = -c(\dot{x} - \dot{y}) - k(x - y) \tag{8.66}$$

Because the instrument actually only senses the relative motion between M and the case, i.e.,

$$z = x - y$$

Equation (8.66) becomes

$$M\ddot{z} + c\dot{z} + kz = -M\ddot{y} \tag{8.67}$$

Assuming that the vibration of ground is in sinusoidal motion, $y = Y \sin \omega t$, we find then the equation

$$M\ddot{z} + c\dot{z} + kz = MY\omega^2 \sin \omega t \tag{8.68}$$

2) Note that the preceding equation is in the same form as Eq. (8.55) except the symbols are changed. Using Eq. (8.57), we find

$$z(t) = \frac{Y\omega^2}{\sqrt{D}} \sin(\omega t - \phi)$$

where

$$\tan \phi = \frac{c\omega/m}{k/m - \omega^2}$$

$$D = (k/M - \omega^2)^2 + (c\omega/M)^2$$

Therefore, the amplitude of vibration detected by the instrument is

$$Z = \frac{Y\omega^2}{\sqrt{(k/m - \omega^2)^2 + (c\omega/M)^2}} \tag{8.69}$$

3) Equation (8.69) can be simplified if we define the following.

Natural frequency of undamped oscillation:

$$\omega_n = \sqrt{k/M} \tag{8.70}$$

Critical damping coefficient:

$$c_c = 2M\omega_n \tag{8.71}$$

Damping factor:

$$\zeta = c/c_c \tag{8.72}$$

Then

$$\frac{c\omega}{M} = \frac{c\omega}{k}\frac{k}{M} = \frac{c}{c_c}\frac{c_c\omega}{k}\frac{k}{M} = \zeta\frac{2M\omega_n\omega}{k}\frac{k}{M} = 2\zeta\omega\omega_n$$

$$Z = \frac{Y(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \tag{8.73}$$

From this equation we can easily see that

$$Z \approx Y$$

if $\omega \gg \omega_n$ and $\zeta \ll 1$. Therefore we always use a large mass, weak spring, and small damper such that $\omega/\omega_n \to \infty$ and $\zeta \to 0$.

4) The Richter scale is defined as

$$R \equiv \log_{10}(Y/Y_s) \tag{8.74}$$

Y_s is the standard magnitude of an earthquake that is 1 μ, or 10^{-6} m. For a Richter scale value of 7

$$\log_{10}(Y/Y_s) = 7$$

$$Y = 10^7 \times 10^{-6}\ \text{m} = 10\ \text{m}$$

Therefore, the amplitude of the earthquake is 10 m.

Example 8.7

An accelerometer is an instrument that directly measures the vibration of a moving object. Then the amplitude of the vibration is converted into acceleration. Once the acceleration is measured, the velocity and displacement can be obtained by integration. The accelerometer is a very important instrument in a submarine. The general construction of the meter consists of a spring, a mass, and a damper similar to the seismometer shown in Fig. 8.10. For this instrument, discuss the essential factors in the design so that the measured quantity truly represents the acceleration.

Solution. Because the construction of the accelerometer is similar to that of the seismometer, Eq. (8.73) is applicable and we have

$$Z = \left(\frac{\omega}{\omega_n}\right)^2 \frac{Y}{f(\omega/\omega_n, \zeta)} \tag{8.75}$$

where

$$f(\omega/\omega_n, \zeta) = \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2} \tag{8.76}$$

In this equation, $Y\omega^2$ is the acceleration, Z is measured, and ω_n^2 is fixed in the design. From this equation, we see that $f(\omega/\omega_n, \zeta) \simeq 1$ as $\omega/\omega_n \ll 1$ and $\varsigma = 0.65$ to 0.70. Therefore in the design of accelerometer, the natural frequency chosen should be very high and the damping factor should be about 0.70.

Example 8.8

Often in engineering practices, vibration of machinery cannot be totally eliminated. To reduce the effect of vibration to adjacent parts, vibration absorber is usually installed between the machine and the supporting ground. The vibration absorber can be modeled by a combination of springs and a damper as shown in Fig. 8.11a.

For the vibration absorber, 1) find the force transmitted from the vibrating mass M to the supporting ground, 2) find the expression for the ratio of the transmitted force and the excitation force, and 3) discuss how to reduce the ratio of forces in part 2.

Solution. 1) Because the present system is similar to the one shown in Fig. 8.8, Eq. (8.55) and its solution can be applied here. Rewrite Eq. (8.57) as

$$x = \mathrm{Im}\{X \exp[i(\omega t - \phi)]\} \tag{8.77}$$

where $X = F_0/(M\sqrt{D})$, then we have

$$\dot{x} = \mathrm{Im}\{X\omega i \exp[i(\omega t - \phi)]\} = \mathrm{Im}\{X\omega \exp[i(\omega t - \phi + \pi/2)]\} \tag{8.78}$$

and

$$\ddot{x} = \mathrm{Im}\{-X\omega^2 \exp[i(\omega t - \phi)]\} = \mathrm{Im}\{X\omega^2 \exp[i(\omega t - \phi + \pi)]\} \tag{8.79}$$

By drawing $F_0 \exp[i\omega t]$, $MX\omega^2 \exp[i(\omega t - \phi + \pi)]$, $c\omega X \exp[i(\omega t - \phi + \pi/2)]$, and $kX \exp[i(\omega t - \phi)]$ on a graph as shown in Fig. 8.11b, we obtain the total force transmitted to the support as

$$F_T = \sqrt{(kX)^2 + (c\omega X)^2} = kX\sqrt{1 + (c\omega/k)^2} \tag{8.80}$$

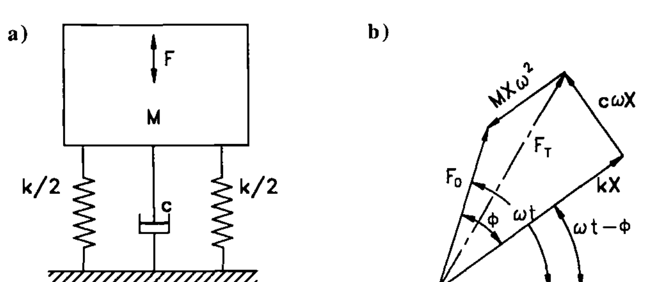

Fig. 8.11 Relation between the vibration system and the supporting ground.

2) Because

$$X = \frac{F_0}{M\sqrt{D}}$$

$$kX = \frac{F_0(k/M)}{\sqrt{(k/M - \omega^2)^2 + (c\omega/M)^2}}$$

$$= \frac{F_0}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + (c\omega/k)^2}}$$

$$= \frac{F_0}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2}}$$

The ratio of the transmitted force to the exciting force is

$$\frac{F_T}{F_0} = \frac{\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2}} \tag{8.81}$$

3) From Eq. (8.81), we easily can see that to reduce the value of the ratio, we set the damping factor ς to zero and choose $\omega/\omega_n \gg 1$. Then we find

$$\frac{F_T}{F_0} = \left(\frac{\omega_n}{\omega}\right)^2 \tag{8.82}$$

Therefore the ratio is reduced if we use weak springs and no damper.

Example 8.9

Figure 8.12 depicts a simplified model of a spring-supported vehicle traveling over a rough road. 1) Find the equation for the amplitude of vibrating mass m, and

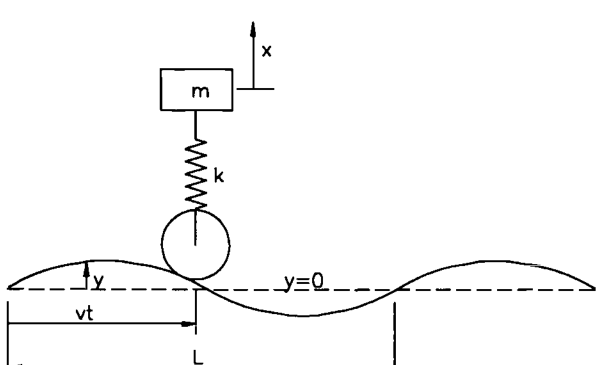

Fig. 8.12 Spring-supported vehicle on rough road.

determine the most unfavorable speed. 2) Suppose that the vehicle is now traveling on a flat road but its tire is cut by a hard braking. Note that this out-of-round wheel is equivalent to driving the vehicle on a road with repeated holes. Determine also the most unfavorable speed.

Solution. 1) For the balance of forces in the vertical direction, we have

$$m\ddot{x} = -k(x - y)$$

or

$$m\ddot{x} + kx = ky$$

The rough road can be modeled as

$$y = Y \sin \omega t$$

where $\omega = 2\pi v/L$. Therefore the equation of motion is

$$m\ddot{x} - kx = kY \sin \omega t \tag{8.83}$$

This equation is similar to Eq. (8.55) except $c = 0$. Hence the solution of Eq. (8.55) can be used. We have

$$X = \frac{kY}{m\sqrt{D}} = \frac{Y}{(\omega/\omega_n)^2 - 1}$$

or

$$\frac{X}{Y} = \frac{1}{(2\pi v/L\omega_n)^2 - 1}$$

The most unfavorable speed occurs when the system is in resonance, i.e.,

$$\begin{aligned} \omega &= \omega_n \\ 2\pi v/L\omega_n &= 1 \\ v &= (L/2\pi)\sqrt{k/m} \end{aligned} \tag{8.84}$$

2) When the tire is out-of-round, the velocity of the vehicle is directly related to ω as

$$v = R\omega$$

The governing equation for the motion still can be written as Eq. (8.83). The condition of resonance is now

$$\begin{aligned} \omega = v/R = \omega_n &= \sqrt{k/m} \\ v &= R\sqrt{k/m} \end{aligned} \tag{8.85}$$

where R is the radius of the tire.

8.5 Transient Vibration

When a spring-mass system is suddenly subjected to a nonperiodic force, a transient vibration is produced, because a steady-state oscillation is usually not present. Let us consider that an impulse is applied to a spring-mass system. The equation for the dynamic behavior of the system can be written as

$$m\ddot{x} + kx = \hat{F}\delta(t - t_1) \tag{8.86}$$

where $\hat{F}$ is an impulse and $\delta(t - t_1)$ is a Dirac delta function with a dimension of t^{-1}. Taking the Laplace transform of the equation, we have

$$m[s^2\bar{x}(s) - sx(0) - \dot{x}(0)] + k\bar{x}(s) = \hat{F}e^{-t_1 s}$$

where $\bar{x}(s)$ is the transformed function of $x(t)$. For $x(0) = \dot{x}(0) = 0$,

$$\bar{x}(s) = \frac{\hat{F}e^{-t_1 s}}{ms^2 + k}$$

Taking the inverse transform by using Eq. (8.43) and the Laplace transform table, we find

$$x(t) = \begin{cases} 0 & \text{as } t < t_1 \\ \frac{\hat{F}}{m\omega_n} \sin\omega_n(t - t_1) & \text{as } t > t_1 \end{cases} \tag{8.87}$$

By introducing a unit step function

$$H(t - t_1) = \begin{cases} 0 & \text{as } t < t_1 \\ 1 & \text{as } t > t_1 \end{cases} \tag{8.88}$$

we can write the solution of Eq. (8.86) as

$$x(t) = \frac{\hat{F}}{m\omega_n} \sin\omega_n(t - t_1)H(t - t_1) \tag{8.89}$$

Example 8.10

A simple spring-mass system is subjected to a repeated impulse $\hat{F}$ of finite duration at intervals of τ as shown in Fig. 8.13. Find the transient response.

Solution. Following Eq. (8.86) we write the equation of motion as

$$m\ddot{x} + kx = \hat{F}\sum_{n=0}^{N} \delta(t - n\tau) \tag{8.90}$$

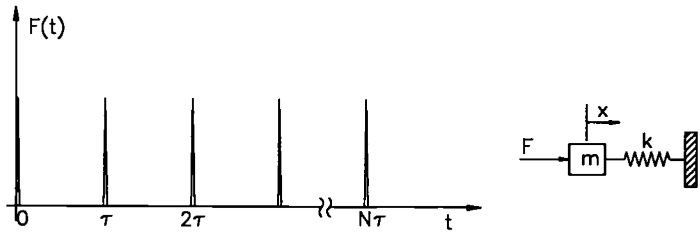

Fig. 8.13 Repeated impulse on a spring-mass system.

where N is a finite integer. Taking the Laplace transform and for $x(0) = \dot{x}(0) = 0$, we have

$$\bar{x}(s) = \frac{\hat{F}}{ms^2 + k} \sum_{n=0}^{N} e^{-n\tau s}$$

Therefore, from the inverse Laplace transform of the preceding equation, we find the transient response

$$x(t) = \sum_{n=0}^{N} \frac{\hat{F}}{m\omega_n} \sin \omega_n (t + n\tau) H(t - n\tau) \tag{8.91}$$

Note that, at time t_1, N is the integer of t_1/τ in the preceding equation because of the unit step functions; if $t_1 > N\tau$, then N is the total number of impulses used in Eq. (8.90). Two particular cases may be discussed here:

1) As $\omega_n \tau = 2\pi$, then

$$\begin{aligned} x(t) &= \sum_{n=0}^{N} \frac{\hat{F}}{m\omega_n} \sin(\omega_n t + n2\pi) H(t - n\tau) \\ &= \sum_{n=0}^{N} \frac{\hat{F}}{m\omega_n} \sin(\omega_n t) H(t - n\tau) \\ &= \frac{N_1 \hat{F}}{m\omega_n} \sin(\omega_n t) \qquad N_1 = \text{integer}(t_1/\tau) \end{aligned} \tag{8.92}$$

This equation shows that the amplitude of $x(t)$ increases with time as indicated by N_1 in the expression. This means that the impulse is in resonance with the system.

2) As $\omega_n \tau = \pi$, then

$$\begin{aligned} x(t) &= \sum_{n=0}^{N} \frac{\hat{F}}{m\omega_n} \sin(\omega_n t - n\pi) H(t - n\tau) \\ &= \sum_{n=0}^{N} \frac{\hat{F}}{m\omega_n} (-1)^n \sin(\omega_n t) H(t - n\tau) \end{aligned} \tag{8.93}$$

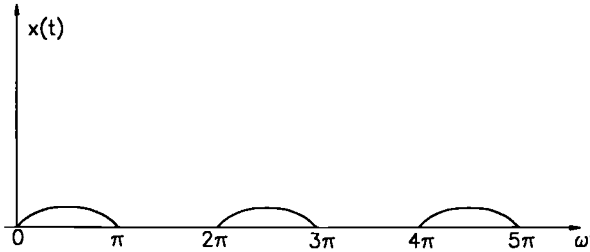

Fig. 8.14 Transient response at minimum amplitude.

The transient response of the preceding expression is shown in Fig. 8.14. If N is odd, the response is 0 as $N\tau < t < (N+1)\tau$; if N is even, the response is

$$\frac{\hat{F}}{m\omega_n}\sin\omega_n t$$

as $N\tau < t < (N+1)\tau$.

Next let us consider that an impulse is applied to a spring-mass system with a damper. Then the equation of motion is

$$m\ddot{x} + c\dot{x} + kx = \hat{F}\delta(t) \tag{8.94}$$

with $x(0) = \dot{x}(0) = 0$. Taking the Laplace transform, we have

$$ms^2\bar{x}(s) + cs\bar{x}(s) + k\bar{x}(s) = \hat{F}$$

or

$$\bar{x}(s) = \frac{\hat{F}}{ms^2 + cs + k}$$

The transient response is the inverse Laplace transform of the equation. By using the table of Laplace Transforms in Appendix F, we find

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left[\frac{\hat{F}}{ms^2 + cs + k}\right] \\ &= \frac{\hat{F}}{m}\mathcal{L}^{-1}\left[\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right] \\ &= \frac{\hat{F}}{m\omega_n\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t) \end{aligned} \tag{8.95}$$

Furthermore, let us consider a general system with an arbitrary excitation. The equation of motion is in the form of

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{8.96}$$

with $x(0) = \dot{x}(0) = 0$. Taking the Laplace transform, we obtain

$$\bar{x}(s) = \frac{\bar{f}(s)}{ms^2 + cs + k} = \frac{\bar{f}(s)}{m} \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Let

$$\frac{1}{m\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \bar{g}(s)$$

then

$$\bar{x}(s) = \bar{f}(s)\bar{g}(s)$$

Using the convolution, Eq. (8.46), and Eq. (8.95), we find

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[\bar{f}(s)\bar{g}(s)] = \int_0^t f(\eta)g(t-\eta)\,\mathrm{d}\eta \\ &= \frac{1}{m\omega_n\sqrt{1-\zeta^2}} \int_0^t f(\eta)e^{-\zeta\omega_n(t-\eta)} \sin[\omega_n\sqrt{1-\zeta^2}(t-\eta)]\,\mathrm{d}\eta \qquad (8.97) \end{aligned}$$

For a special case, $c = 0$ or $\varsigma = 0$, then

$$x(t) = \frac{1}{m\omega_n} \int_0^t f(\eta) \sin[\omega_n(t-\eta)]\,\mathrm{d}\eta \qquad (8.98)$$

Example 8.11

Determine the response of an undamped system with a single degree of freedom subjected to the following excitation:

$$f(t) = \begin{cases} F_0 \sin\omega t & \text{as } 0 < t < \pi/\omega \\ 0 & \text{as } t < 0 \quad \text{and} \quad t > \pi/\omega \end{cases}$$

Solution. Using Eq. (8.98) for $0 < t < \pi/\omega$, we have

$$\begin{aligned} x(t) &= \frac{F_0}{m\omega_n} \int_0^t \sin\omega\eta \sin[\omega_n(t-\eta)]\,\mathrm{d}\eta \\ &= \frac{F_0}{2m\omega_n} \int_0^t \{\cos[(\omega+\omega_n)\eta - \omega_n t] - \cos[(\omega-\omega_n)\eta + \omega_n t]\}\,\mathrm{d}\eta \\ &= \frac{F_0}{k\left[1-(\omega/\omega_n)^2\right]} \left[\sin\omega t - \frac{\omega}{\omega_n}\sin\omega_n t\right] \qquad (8.99) \end{aligned}$$

For $t > \pi/\omega$, we find

$$x(t) = \frac{F_0}{m\omega_n}\int_0^{\frac{\pi}{\omega}} \sin\omega\eta \sin[\omega_n(t-\eta)]\mathrm{d}\eta$$
$$= \frac{F_0\omega_n/\omega}{k\left[1-(\omega_n/\omega)^2\right]}\left[\sin\omega_n t + \sin\omega_n\left(t-\frac{\pi}{\omega}\right)\right] \tag{8.100}$$

Example 8.12

Determine the response of a damped system with a single degree of freedom subjected to the following excitation

$$f(t) = F_0 H(t) = \begin{cases} F_0 & \text{as } t > 0 \\ 0 & \text{as } t < 0 \end{cases}$$

Solution. Using Eq. (8.97) for a damped system, we find

$$x(t) = \frac{-F_0}{m\omega_n\sqrt{1-\zeta^2}}\int_0^t \exp\left[\zeta\omega_n(\eta-t)\right]\sin[\omega_n\sqrt{1-\zeta^2}(\eta-t)]\mathrm{d}\eta$$
$$= \frac{F_0}{m\omega_n^2}\left\{1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}[\zeta\sin(\omega_n\sqrt{1-\zeta^2}t)+\sqrt{1-\zeta^2}\cos(\omega_n\sqrt{1-\zeta^2}t)]\right\}$$

Let $\zeta = \sin\psi$, then $\sqrt{1-\zeta^2} = \cos\psi$; the response becomes

$$x(t) = \frac{F_0}{k}\left\{1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_n\sqrt{1-\zeta^2}t-\psi)\right\} \tag{8.101}$$

where

$$\tan\psi = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

Example 8.13

A mass m is packaged in a box as shown in Fig. 8.15. The box is dropped through height h. Determine the maximum force transmitted to mass m and the required rattle space at the instant of impact when the box reaches the ground. Assume that the impact can be represented by an impulse.

1) Assume the mass of the box is much greater than m, so that the free fall of the box is not influenced by the relative motion of the mass m.

2) On striking the floor, the impact depends greatly on the material properties of the box and the floor. It is reasonable to have the impact represented by an impulse.

3) Assume that the box is rigid. No deformation is considered.

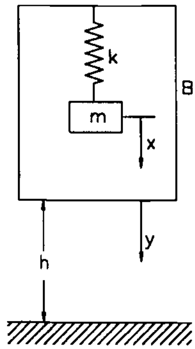

Fig. 8.15 Packaging analysis.

Based on the preceding assumptions, the equation for the system can be written as

$$m(\ddot{x} + \ddot{y}) = -kx$$

$$m\ddot{x} + kx = -m\ddot{y}$$

or

$$\ddot{x} + \omega_n^2 x = -\ddot{y} \tag{8.102}$$

where x is the displacement of mass m from its equilibrium position relative to the box and y is the displacement of the box from its initial position. Taking the Laplace transform of the preceding equation, we find

$$\bar{x}(s) = [x(0) + y(0)]\frac{s}{s^2 + \omega_n^2} + [\dot{x}(0) + \dot{y}(0)]\frac{1}{s^2 + \omega_n^2} - \frac{s^2\bar{y}(s)}{s^2 + \omega_n^2}$$

The response of $x(t)$ is obtained from

$$x(t) = \mathcal{L}^{-1}[\bar{x}(s)]$$

$$= [x(0) + y(0)]\cos\omega_n t + \frac{1}{\omega_n}[\dot{x}(0) + \dot{y}(0)]\sin\omega_n t - \mathcal{L}^{-1}\left[\frac{s^2\bar{y}(s)}{s^2 + \omega_n^2}\right]$$

For the time of the free falling,

$$x(0) = y(0) = \dot{x}(0) = \dot{y}(0) = 0$$

$$y(t) = \tfrac{1}{2}gt^2, \qquad \bar{y}(s) = (g/s^3)$$

$$x(t) = -\mathcal{L}^{-1}\left[\frac{g}{s\left(s^2 + \omega_n^2\right)}\right]$$

$$= -\frac{g}{\omega_n^2}(1 - \cos\omega_n t)$$

At the instant of impact,

$$t_0 = \sqrt{2h/g}$$

$$x(t_0) = -\left(g/\omega_n^2\right)(1 - \cos\omega_n t_0)$$

$$\dot{x}(t_0) = -(g/\omega_n)\sin\omega_n t_0$$

These quantities become the initial conditions for the second phase of the problem after the impact of the box with the floor.

After the impact of the box with the floor, consider $t = 0$ at the instant of impact:

$$x(0) = -\frac{g}{\omega_n^2}(1 - \cos\omega_n t_0)$$

$$\dot{x}(0) = -\frac{g}{\omega_n}\sin\omega_n t_0$$

$$\ddot{y}(t) = -\frac{\hat{F}}{m_b}\delta(t)$$

Therefore $\mathcal{L}[\ddot{y}(t)] = -\hat{F}/m_b$;

$$\bar{x}(s) = x(0)\frac{s}{s^2 + \omega_n^2} + \dot{x}(0)\frac{1}{s^2 + \omega_n^2} + \frac{\hat{F}}{m_b}\frac{1}{s^2 + \omega_n^2}$$

where m_b is the mass of the box. We find the response

$$\begin{aligned} x(t) &= x(0)\cos\omega_n t + \frac{1}{\omega_n}\left[\dot{x}(0) + \frac{\hat{F}}{m_b}\right]\sin\omega_n t \\ &= -\frac{g}{\omega_n^2}(1 - \cos\omega_n t_0)\cos\omega_n t + \frac{1}{\omega_n}\left[\frac{\hat{F}}{m_b} - \frac{g}{\omega_n}\sin\omega_n t_0\right]\sin\omega_n t \\ &= \frac{g}{\omega_n^2}\sqrt{(1 - \cos\omega_n t_0)^2 + \left[\frac{\hat{F}\omega_n}{m_b g} - \sin\omega_n t_0\right]^2}\sin(\omega_n t - \phi) \end{aligned}$$

$$\tan\phi = \frac{1 - \cos\omega_n t_0}{(\hat{F}\omega_n/m_b g) - \sin\omega_n t_0}$$

For a special case, as $(\hat{F}\omega_n/m_b g) \gg 1$,

$$\begin{aligned} x(t) &= \frac{g}{\omega_n^2}\frac{\hat{F}\omega_n}{m_b g}\sin(\omega_n t - \phi) \\ &= \frac{\hat{F}}{m_b\omega_n}\sin(\omega_n t - \phi) \end{aligned}$$

Because $x(t)$ is the displacement of m relative to the box, the maximum space required for m to travel before reaching the wall of the box is

$$x_{\max} = \frac{g}{\omega_n^2}\sqrt{(1-\cos\omega_n t_0)^2 + \left(\frac{\hat{F}\omega_n}{m_b g} - \sin\omega_n t_0\right)^2} \tag{8.103}$$

and the maximum force applied is simply $kx_{\max}$.

8.6 Response Spectrum

A response spectrum is a plot of the maximum peak response as a function of the product of the natural frequency of the oscillator and the characteristic time of the applied force.

From the information revealed in the plot, we can modify the design so that the peak response will be within the expected range. To illustrate the use of response spectrum, let us see the following example.

Example 8.14

Determine the response spectrum for a mass-spring system subjected to a force as a function of time given as follows:

$$\begin{aligned} f(t) &= F_0(t/t_1) && \text{as } 0 < t < t_1 \\ &= F_0 && \text{as } t > t_1 \end{aligned}$$

Solution. For the interval of $0 < t < t_1$, the response is obtained from Eq. (8.98):

$$\begin{aligned} x(t) &= \frac{1}{m\omega_n}\int_0^t \frac{F_0\eta}{t_1}\sin\omega_n(t-\eta)\mathrm{d}\eta \\ &= \frac{F_0}{k}\left[\frac{t}{t_1} - \frac{\sin\omega_n t}{\omega_n t_1}\right] \end{aligned} \tag{8.104}$$

For the time $t_1 < t < \infty$, the response is

$$\begin{aligned} x(t) &= \frac{1}{m\omega_n}\left[\int_0^{t_1}\frac{F_0\eta}{t_1}\sin\omega_n(t-\eta)\,\mathrm{d}\eta + \int_{t_1}^t F_0\sin\omega_n(t-\eta)\mathrm{d}\eta\right] \\ &= \frac{F_0}{k}\left[1 + \frac{1}{\omega_n t_1}\sin\omega_n(t-t_1) - \frac{1}{\omega_n t_1}\sin\omega_n t\right] \end{aligned} \tag{8.105}$$

Examining Eqs. (8.104) and (8.105), we can see easily that the response from Eq. (8.105) is higher than that from Eq. (8.104). Hence the response spectrum is determined from Eq. (8.105). To find maximum $x(t)$, we differentiate $x(t)$ with respect to time and set it to zero and obtain

$$\cos\omega_n(t_p - t_1) - \cos\omega_n t_p = 0 \tag{8.106}$$

where t_p is a particular time such that $\dot{x}(t) = 0$, as $t = t_p$. Recalling that

$$\cos\alpha - \cos\beta = -2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta)$$

we conclude that t_p must satisfy the equation

$$\sin\omega_n\left(t_p - \frac{t_1}{2}\right)\sin\omega_n\frac{t_1}{2} = 0$$

which yields the solution

$$t_p = \frac{n\pi}{\omega_n} + \frac{t_1}{2} \qquad n = 1, 2, 3, \ldots$$

or

$$\omega_n(t_p - t_1) = 2n\pi - \omega_n t_p \tag{8.107}$$

From Eq. (8.106), we also find

$$\tan\omega_n t_p = \frac{1-\cos\omega_n t_1}{\sin\omega_n t_1}$$

$$\sin\omega_n t_p = -\sqrt{\frac{1}{2}(1-\cos\omega_n t_1)} \tag{8.108}$$

Using Eq. (8.107), we obtain

$$\sin\omega_n(t_p - t_1) = \sin(-\omega_n t_p) = -\sin\omega_n t_p \tag{8.109}$$

Then, the peak amplitude is found as

$$x_{\max} = \frac{F_0}{k}\left[1 + \frac{1}{\omega_n t_1}\sqrt{2(1-\cos\omega_n t_1)}\right] \tag{8.110}$$

The response spectrum is plotted in Fig. 8.16, which shows that $kx_{\max}/F_0 \to 1$ as $\omega_n t_1$ approaches infinity. Therefore, if the desired response is small, the natural frequency should be chosen as high as possible; otherwise set at

$$\omega_n t_1 = 2n\pi \qquad n = 1, 2, 3, \ldots$$

A velocity spectrum is a plot of the maximum velocity of the mass m versus time for a single-degree-of-freedom oscillator. It is often used for analyses of earthquakes or other ground shock situations. With the formulation of the seismometer given in Example 8.4, we can write

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\ddot{y} \tag{8.111}$$

where $z = x - y$ is the relative displacement between the mass m and the coil

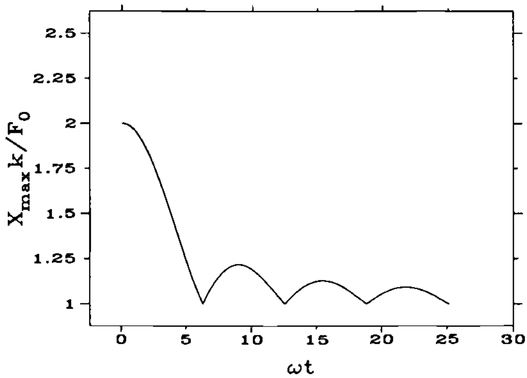

Fig. 8.16 Response spectrum.

attached to the ground. Assuming $z(0) = \dot{z}(0) = 0$ and using Eq. (8.97), we have

$$z(t) = \frac{-1}{\omega_n\sqrt{1-\zeta^2}} \int_0^t \ddot{y}(\eta)\exp[-\zeta\omega_n(t-\eta)]\sin\sqrt{1-\zeta^2}\omega_n(t-\eta)\,\mathrm{d}\eta$$

Recalling the differential formula

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\phi(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_{A(x)}^{B(x)} f(x,t)\,\mathrm{d}t \\ &= \int_A^B \frac{\partial f(x,t)}{\partial x}\mathrm{d}t + f(x,B)\frac{\mathrm{d}B}{\mathrm{d}x} - f(x,A)\frac{\mathrm{d}A}{\mathrm{d}x}\end{aligned} \tag{8.112}$$

we find

$$\dot{z}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t,\eta)\,\mathrm{d}\eta = \int_0^t \frac{\partial f(t,\eta)}{\partial t}\mathrm{d}\eta + f(t,t)$$

or

$$\begin{aligned}\dot{z}(t) = \frac{-1}{\omega_n\sqrt{1-\zeta^2}} \int_0^t \ddot{y}(\eta)\exp[-\zeta\omega_n(t-\eta)][-\zeta\omega_n\sin\sqrt{1-\zeta^2}\omega_n(t-\eta) \\ + \omega_n\sqrt{1-\zeta^2}\cos\sqrt{1-\zeta^2}\omega_n(t-\eta)]\,\mathrm{d}\eta\end{aligned} \tag{8.113}$$

Let

$$A = \int_0^t \ddot{y}(\eta)e^{\zeta\omega_n\eta}\cos\sqrt{1-\zeta^2}\omega_n\eta\,\mathrm{d}\eta$$

$$B = \int_0^t \ddot{y}(\eta)e^{\zeta\omega_n\eta}\sin\sqrt{1-\zeta^2}\omega_n\eta\,\mathrm{d}\eta$$

Equation (8.113) can be written as

$$\dot{z}(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}[(A\zeta - B\sqrt{1-\zeta^2})\sin\sqrt{1-\zeta^2}\omega_n t$$
$$- (A\sqrt{1-\zeta^2} + B\zeta)\cos\sqrt{1-\zeta^2}\omega_n t]$$

or

$$\dot{z}(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sqrt{A^2+B^2}\sin(\sqrt{1-\zeta^2}\omega_n t - \phi) \tag{8.114}$$

$$\tan\phi = \frac{B\zeta + A\sqrt{1-\zeta^2}}{A\zeta - B\sqrt{1-\zeta^2}} \tag{8.115}$$

If $\dot{z}(t)$ is plotted against time, it would appear as an amplitude modulated wave. Because

$$|\dot{z}(t)|_{\max} = \left|\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sqrt{A^2+B^2}\right|_{\max} \tag{8.116}$$

the envelope of the amplitude modulated wave is the velocity spectrum. Two terms also may be mentioned here: pseudo-response spectrum is the plot of

$$|z|_{\max} = \frac{1}{\omega_n}|\dot{z}(t)|_{\max}$$

and pseudo-acceleration spectrum is the plot of

$$|\ddot{z}|_{\max} = \omega_n|\dot{z}(t)|_{\max}$$

8.7 Applications of Fourier Transforms

The Fourier transform was introduced in Section 8.2. In this section we shall study its applications. The Fourier transform of a function $f(t)$ can be written as

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} e^{-iut} f(t)\,\mathrm{d}t \tag{8.117}$$

and the inverse transform of $\mathcal{F}(u)$ is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iut}\mathcal{F}(u)\,\mathrm{d}u \tag{8.118}$$

In these equations, the dimension of u is 1/time, which is the same as the frequency dimension. This means that $\mathcal{F}(u)$ is the function in the frequency domain. If the function $f(t)$ is the amplitude of vibration as a function of time, then, after taking the Fourier transform, we obtain $\mathcal{F}(u)$, which is the amplitude of vibration as a function of frequency. When the result of $\mathcal{F}(u)$ is displayed, we can see the amplitudes of vibration at different frequencies. From this result, we may find the source of vibration and devise a way to suppress it. Therefore, the Fourier transform is useful for analyzing vibration. We will present more details of this application later in this section.

Another application of the Fourier transform is that it can be used to solve differential equations as shown in the following example.

Example 8.15

For an undamped vibration system with a single degree of freedom, apply the Fourier transform to determine the response $x(t)$ subjected to the excitation given below:

$$f(t) = \begin{cases} F_0 & \text{for } -T < t < T \\ 0 & \text{for } t < -T \quad \text{and} \quad t > T \end{cases}$$

Solution. Taking the Fourier transform of the equation

$$m\ddot{x} + kx = f(t)$$

or

$$\ddot{x} + \omega_n^2 x = (1/m)f(t)$$

with $\dot{x}(0) = x(0) = 0$, we obtain

$$(iu)^2\bar{x}(u) + \omega_n^2\bar{x}(u) = (1/m)\mathcal{F}(u) \tag{8.119}$$

where $\bar{x}(u)$ is the Fourier transform of $x(t)$, and

$$\mathcal{F}(u) = \int_{-T}^{T} F_0 e^{-iut}\mathrm{d}t = \frac{F_0}{iu}(e^{iuT} - e^{-iuT})$$

Hence, Eq. (8.119) becomes

$$\bar{x}(u) = \frac{F_0}{k}\frac{1}{iu\left[1 - (u/\omega_n)^2\right]}(e^{iuT} - e^{-iuT}) \tag{8.120}$$

The response $x(t)$ is the inverse transform of $\bar{x}(u)$, i.e.,

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{x}(u)e^{iut}\mathrm{d}u \\ &= \frac{F_0}{k}\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{iuT} - e^{-iuT}}{u\left[1 - (u/\omega_n)^2\right]}e^{iut}\mathrm{d}u \end{aligned}$$

Using the expansion of partial fractions

$$\frac{1}{u\left[1-(u/\omega_n)^2\right]}=\frac{1}{u}-\frac{1}{2(u-\omega_n)}-\frac{1}{2(u+\omega_n)}$$

we have

$$x(t)=\frac{F_0}{k2\pi i}\int_{-\infty}^{\infty}\left[\frac{1}{u}-\frac{1}{2(u-\omega_n)}-\frac{1}{2(u+\omega_n)}\right]$$
$$\cdot\{\exp[iu(t+T)]-\exp[iu(t-T)]\}\,\mathrm{d}u \tag{8.121}$$

To evaluate the preceding integrals, we must perform contour integrations in the complex plane as given in Appendix G. We present the results here; namely,

$$\int_{-\infty}^{\infty}\frac{e^{iu\lambda}}{u}\mathrm{d}u=\begin{cases}0 & \text{as } \lambda<0\\ 2\pi i & \text{as } \lambda>0\end{cases}$$

$$\int_{-\infty}^{\infty}\frac{e^{iu\lambda}}{u-\omega_n}\mathrm{d}u=\begin{cases}0 & \text{as } \lambda<0\\ 2\pi i e^{i\omega_n\lambda} & \text{as } \lambda>0\end{cases}$$

$$\int_{-\infty}^{\infty}\frac{e^{iu\lambda}}{u+\omega_n}\mathrm{d}u=\begin{cases}0 & \text{as } \lambda<0\\ 2\pi i e^{-i\omega_n\lambda} & \text{as } \lambda>0\end{cases}$$

Note that $\lambda=t+T$ or $t-T$, when the preceding results are used for Eq. (8.121), we find

$$x(t)=0 \qquad \text{as } t<-T \tag{8.122}$$

$$x(t)=\frac{F_0}{k}\frac{1}{2\pi i}\left\{2\pi i-\frac{1}{2}(2\pi i)\exp[i\omega_n(t+T)]-\frac{1}{2}(2\pi i)\exp[-i\omega_n(t+T)\}\right]$$
$$=\frac{F_0}{k}[1-\cos\omega_n(t+T)] \qquad \text{as } -T<t<T \tag{8.123}$$

$$x(t)=\frac{F_0}{k}\frac{1}{2\pi i}\left\{\left[2\pi i-\frac{1}{2}(2\pi i)\exp[i\omega_n(t+T)]-\frac{1}{2}(2\pi i)\exp[-i\omega_n(t+T)]\right]\right.$$
$$\left.-\left[2\pi i-\frac{1}{2}(2\pi i)\exp[i\omega_n(t-T)]-\frac{1}{2}(2\pi i)\exp[-i\omega_n(t-T)]\right]\right\}$$
$$=\frac{F_0}{k}[\cos\omega_n(t-T)-\cos\omega_n(t+T)] \qquad \text{as } t>T \tag{8.124}$$

In analyzing vibration in the field, because of limited time, samples of data are taken in a finite interval of T_0. The samples are assumed to be a periodic function with period of T_0 and with N points in the interval. Hence, the samples can be represented as

$$f(t)\delta(t-kT) \quad k=0,1,2,\ldots,N-1$$

Note that $\delta(t - kT)$ is assumed to be dimensionless. Because of the characteristics of the function $f(t)\delta(t - kT)$, its Fourier transform cannot be performed in the usual manner. To satisfy the conditions for Fourier transform as stated previously, the transform of this function is performed approximately in discrete manner. Thus the transform is known as the discrete Fourier transform. Consider the Fourier transform

$$\mathcal{F}(n) = \int_{-\infty}^{\infty} \exp\left[-\frac{i2\pi nt}{T_0}\right] f(t)\,\mathrm{d}t$$

in which the frequency u has been replaced by $2\pi n/T_0$. The interval between datum points is T, which is also assumed as the interval of the delta function. Thus $NT = T_0$. Therefore, the discrete Fourier transform of $f(t)\delta(t - kT)$ is performed as

$$\begin{aligned}\mathcal{F}(n) &= \sum_{k=0}^{N-1} \int_{-\frac{T}{2}}^{T_0+\frac{T}{2}} f(t)\delta(t - kT) \exp\left[-i\frac{2\pi nt}{T_0}\right]\mathrm{d}t \\ &= T\sum_{k=0}^{n-1} f(kT)e^{-i2\pi nkT/T_0} \\ &= T\sum_{k=0}^{n-1} f(kT)e^{-i2\pi nk/N} \qquad n = 0, 1, \ldots, N-1 \end{aligned} \tag{8.125}$$

Similarly, the corresponding discrete inverse Fourier transform is manipulated as follows:

$$\begin{aligned} f(kT) &= \int_{-\infty}^{\infty} \mathcal{F}(f)e^{i2\pi kTf}\mathrm{d}f \\ &= \sum_{n=0}^{N-1} \int_{-\frac{1}{NT}}^{\frac{N+1}{NT}} \mathcal{F}(f)\delta(f - n\Delta f)e^{i2\pi kTf}\mathrm{d}f \\ &= \frac{1}{NT}\sum_{n=0}^{N-1} \mathcal{F}(n\Delta f)e^{i2\pi kTn\Delta f} \\ &= \frac{1}{NT}\sum_{n=0}^{N-1} \mathcal{F}(n)e^{i2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1 \end{aligned} \tag{8.126}$$

Note that in the preceding derivation, the width of the Dirac delta function is Δf, (i.e., $1/NT = \Delta f$, or $T\Delta f = 1/N$). Further, the data of $\mathcal{F}(f)$ available are only N points in the range of f from 0 to $1/T$; outside of the range are assumed to be zero. Also notice that Δf is omitted in the last expression of the transformed function $\mathcal{F}(n)$.

From Eq. (8.125), the vibration amplitude in the frequency domain is determined. Because rotating speeds of moving parts are usually known, their frequencies can be calculated. Once the frequencies of vibration are known, the source of vibration can be determined. Consequently any undesirable vibration may be eliminated.

Problems

8.1. Determine the Fourier series for the rectangular pulses given here:

$$f(x) = 1 \qquad \text{as } 0 < x < \pi$$

$$f(x) = 0 \qquad \text{as } \pi < x < 2\pi$$

$$f(x) = f(x + 2\pi)$$

8.2. Find the Fourier integral representations for the function:

$$f(x) = x \qquad \text{as } 0 < x < a$$

$$= 0 \qquad \text{as } x > a$$

with the two different conditions: $f(x) = f(-x)$ and $f(-x) = -f(x)$.

8.3. Expand the following function in a Fourier series of period 2π:

$$f(t) = \begin{cases} 0 & \text{as } -\pi < t < 0 \\ t(\pi - t) & \text{as } 0 < t < \pi \end{cases}$$

8.4. Find Fourier transforms of the derivatives of a function: $f'(x)$ and $f''(x)$.

8.5. Determine the Laplace transforms of the two following functions: $f(t) = \sin \omega t$ and $f(t) = a(1 - e^{-t})$, where a and ω are constant.

8.6. A cylinder of mass m and mass moment of inertia J is free to roll without slipping but is restrained by a spring k (Fig. P8.6).

(a) Find the equation of motion and determine the natural frequency of oscillation.

(b) Suppose that the cylinder is pulled away from its equilibrium position horizontally by distance a and is released at rest. Find the response of the system.

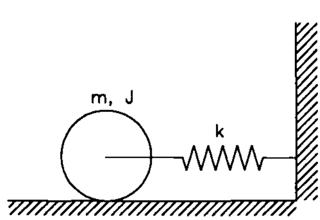

Fig. P8.6

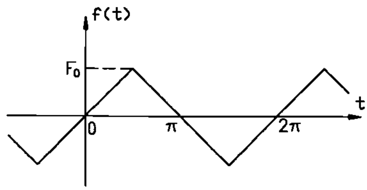

Fig. P8.8

8.7. A damped spring-mass system is started in oscillation under the initial conditions: $x = 0, \dot{x} = v_0$.

(a) Determine the equations of motion when 1) $\zeta = 0.5$, 2) $\zeta = 1.0$, and 3) $\zeta = 2.0$.

(b) Find the responses for the three cases.

8.8. If the periodic force shown in Fig. P8.8 is applied to an undamped spring-mass system, determine the responses of the system subjected to various harmonics.

8.9. The system shown in Fig. P8.9 models a vehicle with an out-of-round tire traveling on a flat ground. For a constant vehicle speed v, find the equation of motion and the steady-state solution, and obtain the speed under the resonance condition.

8.10. The differential equation of motion for a certain undamped spring-mass system is $4\ddot{x} + 14400x = p(t)$, where the forcing function $p(t)$ is defined by Fig. P8.10. The initial conditions are $x = \dot{x} = 0$. Determine analytically the displacement x for the range of time from 0 to 0.2.

8.11. An undamped spring-mass system is excited by a force $F = te^{-t}$. Assume that the initial conditions are $x = \dot{x} = 0$. Determine the response using Laplace transforms.

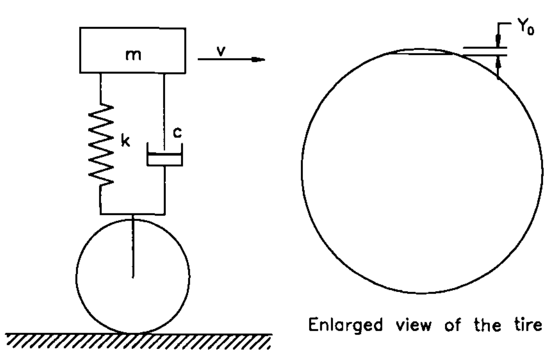

Fig. P8.9

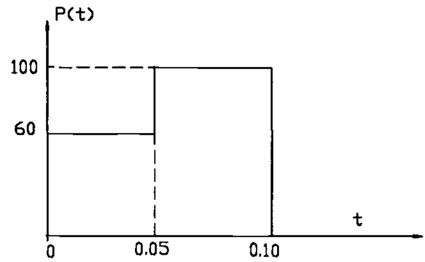

Fig. P8.10

8.12. Determine the response spectrum for an undamped spring-mass system subjected to a rectangular pulse given by

$$f(t) = \begin{cases} F_0 & \text{as } 0 < t < t_0 \\ 0 & \text{as } t > t_0 \end{cases}$$

8.13. The support of a simple pendulum is given a harmonic motion $y_0 = Y_0 \sin \omega t$ along a vertical line, as shown in Fig. P8.13. Find the equation of motion for the system under a small amplitude of oscillation. Determine the steady-state solution.

8.14. Consider the system shown in Fig. P8.14, where the displacements of masses m_1 and m_2 are x_1 and x_2 measured from fixed reference positions, and the amount of the streching of the spring is given by

$$x = x_1 - x_2$$

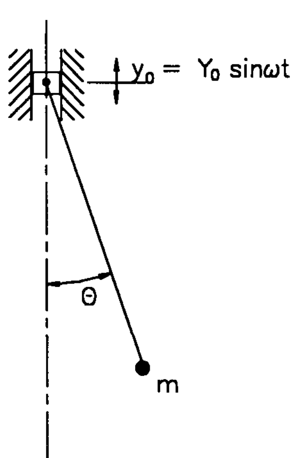

Fig. P8.13

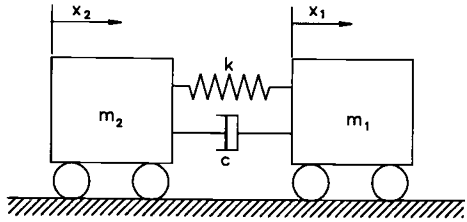

Fig. P8.14

Assuming no friction between the masses and the support and arbitrary initial conditions, find the equation of motion for the system and the response $x(t)$.

9 Vibration of Systems with Multiple Degrees of Freedom

In Chapter 8, we studied linear vibration systems with one degree of freedom and mathematical methods that are fundamental for analyzing these problems. In this chapter, we will study linear and nonlinear systems with multiple degrees of freedom and relatively advanced mathematical techniques for dealing with them.

Section 9.1 deals with various types of vibration systems with two degrees of freedom. There are five examples to illustrate different methods of formulating and analyzing them.

In Section 9.2, we will study vibration systems with multiple degrees of freedom. Because a system with n degrees of freedom is associated with n differential equations, one way to deal with it is to apply matrix methods. With the fundamentals of matrix introduced in Chapter 5, this section may be considered as the additional application of the matrix. Readers will see that there are many advantages with this formulation.

In Section 9.3, we will present the method of lumped parameters with transfer matrices for modeling a vibration system. They may be considered as approximations for modeling the continuous system. The advantage of this approach is that the governing equation can be formulated by the method of transfer matrices, and frequencies and shapes of principal modes can be determined without solving the equations completely. Furthermore, the result of this method can be used to check the results solved from the partial differential equations for a continuous system.

Section 9.4 covers the vibration of continuous systems, which include vibrating string, beam, membrane, and sound wave. Governing equations for these systems are known as wave equations. The use of Fourier series for periodic functions is illustrated repeatedly. Wave equations for one-dimensional space in rectangular, cylindrical, and spherical coordinates are all considered. We notice that the wave form remains the same in rectangular coordinates as the wave propagates either in the positive x or negative x direction. The wave form decays in the cylindrical coordinates because of the properties of Bessel functions. In the spherically symmetric wave, the amplitude decays inversely proportionally to the distance from the center of the wave. From these, the reader can learn some fundamentals in the formulation of the equations and in the determination of solutions.

Section 9.5 is devoted specially to nonlinear systems. As we know from mathematics, a systematic method for solving nonlinear problems is the small perturbation method, which has been introduced in Chapter 5 and is not to be repeated here. Of course, many nonlinear problems can be solved with today's powerful computers. The Runge–Kutta method, which is presented in Appendix A, is a useful tool for obtaining the numerical solutions. However, the disadvantage of numerical method is that it cannot show explicitly the parameters involved in the solutions.

Stability analysis is specially important for nonlinear systems and is presented in Section 9.6. From this section, the reader will find some fundamentals for this subject.

9.1 Vibration Systems with Two Degrees of Freedom

A vibration system with two degrees of freedom requires two spatial coordinates to describe its motion. Consequently, there are two governing equations for the motion and two natural frequencies of vibration. When the system is in a force-free vibration, it vibrates, usually at the combination of two normal modes corresponding to the natural frequencies. However, under forced harmonic vibration, the system will vibrate at the frequency of the excitation in addition to natural frequencies. Resonance will take place if the exciting frequency is the same as one of two natural frequencies. Details of these different situations will be illustrated in the following examples.

Example 9.1

Consider the undamped system as shown in Fig. 9.1. Coordinates x_1 and x_2 are the displacements of m_1 and m_2 away from their equilibrium positions, respectively. Formulate the governing equations of the motion; find the natural frequencies and the steady-state solutions.

Solution. The governing equations may be obtained from the balance of forces. They can be obtained also from Lagrange's equations. Let us take Lagrange's approach. It is seen easily that for the system, kinetic energy is

$$T = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2$$

potential energy is

$$V = \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}k(x_1 - x_2)^2 + \tfrac{1}{2}kx_2^2$$

and Lagrange's function is

$$L = T - V$$

Hence, the equation for x_1 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = m_1\ddot{x}_1 + kx_1 + k(x_1 - x_2) = 0 \tag{9.1}$$

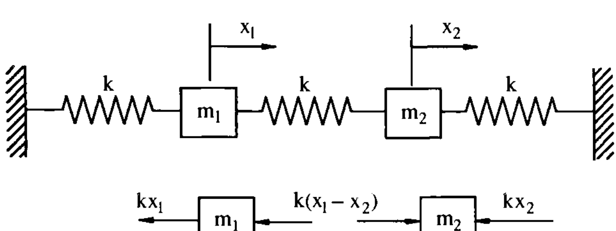

Fig. 9.1 Undamped mass-spring system with two degrees of freedom.

Similarly, for x_2, the equation is

$$m_2\ddot{x}_2 - k(x_1 - x_2) + kx_2 = 0 \tag{9.2}$$

Equations (9.1) and (9.2) are linear second-order differential equations with constant coefficients. The steady-state solution can be assumed as

$$x_1 = A_1 e^{i\omega t}$$
$$x_2 = A_2 e^{i\omega t}$$

Substituting these into the governing equations gives

$$\begin{aligned}(2k - \omega^2 m_1)A_1 - kA_2 &= 0\\ -kA_1 + (2k - \omega^2 m_2)A_2 &= 0\end{aligned} \tag{9.3}$$

Because A_1 and A_2 are not zero, the determinant of the coefficients must be zero, i.e.,

$$\begin{vmatrix}(2k - \omega^2 m_1) & -k\\ -k & (2k - \omega^2 m_2)\end{vmatrix} = 0$$

To save some writing, let us change the symbol ω^2 to λ, then the preceding determinant leads to the characteristic equation

$$\lambda^2 - \frac{m_1 + m_2}{m_1 m_2} 2k\lambda + \frac{3k^2}{m_1 m_2} = 0 \tag{9.4}$$

The two roots of the equation are

$$\left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\} = \frac{k}{m_1 m_2}\left[(m_1 + m_2) \mp \sqrt{(m_1 - m_2)^2 + m_1 m_2}\right]$$

Therefore, the natural frequencies of the system are found to be

$$\left.\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right\} = \left\{\frac{k}{m_1 m_2}\left[(m_1 + m_2) \mp \sqrt{(m_1 - m_2)^2 + m_1 m_2}\right]\right\}^{\frac{1}{2}} \tag{9.5}$$

Because there are two natural frequencies, the steady-state solution can be written as

$$x_j = \mathrm{Re}\left[A_{j1} e^{i\omega_1 t} + A_{j2} e^{i\omega_2 t}\right] \qquad j = 1, 2$$

where A_{j1} and A_{j2} are arbitrary complex coefficients. Without losing generality, we write explicitly the steady-state solution as

$$x_j = a_j \cos\omega_1 t + b_j \sin\omega_1 t + c_j \cos\omega_2 t + d_j \sin\omega_2 t \quad j = 1, 2 \tag{9.6}$$

where a_j, b_j, c_j, and $d_j (j = 1, 2)$ are real arbitrary constants to be determined. By using the initial conditions $x_j(0)$ and $x_j(0)$, we have

$$x_j(0) = a_j + c_j \qquad j = 1, 2 \tag{9.7}$$

$$\dot{x}_j(0) = \omega_1 b_j + \omega_2 d_j \quad j = 1, 2 \tag{9.8}$$

Note that Eq. (9.3) is valid for each mode of the vibration. When Eq. (9.6) is substituted into Eqs. (9.1) and (9.2) for the first normal mode, the coefficients of $\cos \omega_1 t$ and $\sin \omega_1 t$ must be zero. Hence we find

$$\left(2 - \omega_1^2 \frac{m_1}{k}\right) a_1 - a_2 = 0$$

$$\left(2 - \omega_1^2 \frac{m_1}{k}\right) b_1 - b_2 = 0$$

or

$$k_1 a_1 - a_2 = 0, \qquad k_1 b_1 - b_2 = 0 \tag{9.9}$$

where $k_1 = 2 - \omega_1^2 (m_1/k)$. Similarly, for the second normal mode we have

$$k_2 c_2 - c_1 = 0, \qquad k_2 d_2 - d_1 = 0 \tag{9.10}$$

where $k_2 = 2 - \omega_2^2 (m_2/k)$. From Eqs. (9.7–9.10), a_j, b_j, c_j, and d_j are determined. The results are written as follows:

$$a_1 = \frac{1}{1 - k_1 k_2}[x_1(0) - k_2 x_2(0)] \qquad a_2 = k_1 a_1 \tag{9.11}$$

$$b_1 = \frac{1}{(1 - k_1 k_2)\omega_1}[\dot{x}_1(0) - k_2 \dot{x}_2(0)] \qquad b_2 = k_1 b_1 \tag{9.12}$$

$$c_1 = \frac{-k_2}{1 - k_1 k_2}[k_1 x_1(0) - x_2(0)] \qquad c_2 = c_1/k_2 \tag{9.13}$$

$$d_1 = \frac{-k_2}{(1 - k_1 k_2)\omega_2}[k_1 \dot{x}_1(0) - \dot{x}_2(0)] \qquad d_2 = d_1/k_2 \tag{9.14}$$

Substituting Eqs. (9.11–9.14) into Eq. (9.6) gives the steady-state solutions of $x_1(t)$ and $x_2(t)$. From these we can see that in general the system is vibrating at the combination of two normal modes. To simplify the equations, let us consider a special case, i.e., $m_1 = m_2 = m$. Then the two natural frequencies are

$$\omega_1 = \sqrt{k/m}, \qquad \omega_2 = \sqrt{3k/m} \tag{9.15}$$

The constants are found to be

$$k_1 = 1, \qquad k_2 = -1$$

$$a_1 = \frac{1}{2}[x_1(0) + x_2(0)] = a_2$$

$$b_1 = \frac{1}{2\omega_1}[\dot{x}_1(0) + \dot{x}_2(0)] = b_2$$

$$c_1 = \frac{1}{2}[x_1(0) - x_2(0)] = -c_2$$

$$d_1 = \frac{1}{2\omega_2}[\dot{x}_1(0) - \dot{x}_2(0)] = -d_2$$

Hence the steady-state solutions are

$$x_1(t) = \frac{1}{2}[x_1(0) + x_2(0)]\cos\omega_1 t + \frac{1}{2\omega_1}[\dot{x}_1(0) + \dot{x}_2(0)]\sin\omega_1 t + \frac{1}{2}[x_1(0) - x_2(0)]\cos\omega_2 t + \frac{1}{2\omega_2}[\dot{x}_1(0) - \dot{x}_2(0)]\sin\omega_2 t \tag{9.16}$$

$$x_2(t) = \frac{1}{2}[x_1(0) + x_2(0)]\cos\omega_1 t + \frac{1}{2\omega_1}[\dot{x}_1(0) + \dot{x}_2(0)]\sin\omega_1 t - \frac{1}{2}[x_1(0) - x_2(0)]\cos\omega_2 t - \frac{1}{2\omega_2}[\dot{x}_1(0) - \dot{x}_2(0)]\sin\omega_2 t \tag{9.17}$$

Using these equations, we can see that it is possible for the system to oscillate at a particular frequency. If $x_1(0) = x_2(0)$ and $\dot{x}_1(0) = \dot{x}_2(0)$, the system will vibrate at the first normal mode. On the other hand if $x_1(0) = -x_2(0)$ and $\dot{x}_1(0) = -\dot{x}_2(0)$, then the system vibrates at the second normal mode. However, these conditions are hard to produce in the real world. Therefore, in general the vibration is a combination of two modes.

Through this example, a few remarks shall be made here. Note that the system can vibrate at one of the natural frequencies. The lower frequency is called the fundamental frequency, and the corresponding mode is the fundamental mode. The values of λ_i are called eigenvalues of the characteristic equation. The corresponding ratios of a_2/a_1 and c_2/c_1 or b_2/b_1 and d_2/d_1 obtained from Eqs. (9.9) and (9.10) are the component ratios of eigenvectors. In this example, ω_1 and ω_2 are different. A special case for $\omega_1 = \omega_2$ will be discussed later.

Example 9.2

Consider a torsional system with two degrees of freedom as shown in Fig. 9.2. Assume that the disks have mass moments of inertia of J_1 and J_2 with respect to the rotation axis. θ_1 and θ_2 are the angular displacements of the disks from their equilibrium positions, respectively. The torsional stiffness for the portion of

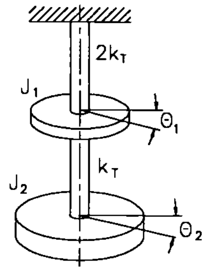

Fig. 9.2 Torsional system with two degrees of freedom.

the shaft between the disks can be expressed as $k_T = GJ/\ell$ where G is the shear modular of elasticity, J is the torsional constant of the cross section, and ℓ is length of the shaft. The torsional stiffness for the portion of the shaft between the support and the first disk is $2k_T$. Formulate the equations of motion, and determine the natural frequencies and shapes of the principal modes.

Solution. This is a conservative system. The kinetic and potential energies are written as

$$T = \tfrac{1}{2}J_1\dot{\theta}_1^2 + \tfrac{1}{2}J_2\dot{\theta}_2^2$$

$$V = \tfrac{1}{2}(2k_T)\theta_1^2 + \tfrac{1}{2}k_T(\theta_1 - \theta_2)^2$$

Lagrange's function is $L = T - V$. The equations of motion are then

$$J_1\ddot{\theta}_1 + 2k_T\theta_1 + k_T(\theta_1 - \theta_2) = 0 \tag{9.18}$$

$$J_2\ddot{\theta}_2 - k_T(\theta_1 - \theta_2) = 0 \tag{9.19}$$

By assuming

$$\theta_i = A_i e^{i\omega t}$$

we have

$$(3k_T/J_1 - \omega^2)A_1 - (k_T/J_1)A_2 = 0 \tag{9.20}$$

$$(-k_T/J_2)A_1 + (k_T/J_2 - \omega^2)A_2 = 0 \tag{9.21}$$

Because A_1 and A_2 cannot be all zero, the determinant of the coefficients must be zero, i.e.,

$$\begin{vmatrix} (3k_T/J_1 - \omega^2) & (-k_T/J_1) \\ (-k_T/J_2) & (k_T/J_2 - \omega^2) \end{vmatrix} = 0$$

Expanding the determinant leads to the characteristic equation

$$\omega^4 - k_T(3/J_1 + 1/J_2)\omega^2 + 2k_T^2/(J_1J_2) = 0 \tag{9.22}$$

The roots are

$$\omega_1^2 = \frac{k_T}{2}\left[(3/J_1 + 1/J_2) - \sqrt{(3/J_1)^2 - (2/J_1 J_2) + (1/J_2)^2}\right] \quad (9.23a)$$

$$\omega_2^2 = \frac{k_T}{2}\left[(3/J_1 + 1/J_2) + \sqrt{(3/J_1)^2 - (2/J_1 J_2) + (1/J_2)^2}\right] \quad (9.23b)$$

Substituting these into Eq. (9.20) gives the modes as

$$\left(\frac{A_1}{A_2}\right)_1 = \frac{k_T/J_1}{3k_T/J_1 - \omega_1^2} \quad (9.24a)$$

$$\left(\frac{A_1}{A_2}\right)_2 = \frac{k_T/J_1}{3k_T/J_1 - \omega_2^2} \quad (9.24b)$$

Example 9.3

Consider the vibration of an automobile modeled as a two-degree-of-freedom system, as shown in Fig. 9.3. The numerical values of the parameters are given as follows:

$$m = 1460\,\text{kg}, \qquad \ell_1 = 1.37\,\text{m}, \qquad \ell_2 = 1.68\,\text{m}$$

$$k_1 = 35\,\text{kN/m}, \qquad k_2 = 38\,\text{kN/m}, \qquad I_C = 2170\,\text{kg-m}^2$$

Determine the natural frequencies and the amplitude ratios under the normal modes of vibration.

Solution. To find the equations of motion, we take Lagrange's approach. Choose x and θ as the generalized coordinates where x is the vertical displacement of the center of mass and θ is the angular displacement of the automobile from the equilibrium position. Then the system will have kinetic energy

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}I_c\dot{\theta}^2$$

and potential energy

$$V = \tfrac{1}{2}k_1(x - \ell_1\theta)^2 + \tfrac{1}{2}k_2(x + \ell_2\theta)^2$$

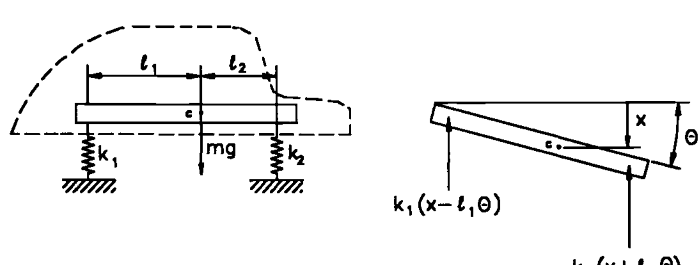

Fig. 9.3 Simplified model for a vibrating automobile.

With the use of Lagrange's equation, we obtain

$$m\ddot{x} + (k_1 + k_2)x + (k_2\ell_2 - k_1\ell_1)\theta = 0 \tag{9.25}$$

$$I_c\ddot{\theta} + (k_2\ell_2 - k_1\ell_1)x + \left(k_1\ell_1^2 + k_2\ell_2^2\right)\theta = 0 \tag{9.26}$$

The preceding equations are statically coupled because there are angular displacement θ terms in the equation for translational motion, Eq. (9.25), and translational displacement x terms in the equation for rotational motion, Eq. (9.26). Note that Eqs. (9.25) and (9.26) are linear ordinary differential equation with constant coefficients. The steady-state solutions can be assumed as

$$x(t) = Xe^{i\omega t}$$

$$\theta(t) = \Theta e^{i\omega t}$$

By substituting these solutions in Eqs. (9.25) and (9.26), we have

$$\begin{pmatrix} (k_1 + k_2 - \omega^2 m) & -(k_1\ell_1 - k_2\ell_2) \\ -(k_1\ell_1 - k_2\ell_2) & \left(k_1\ell_1^2 + k_2\ell_2^2 - \omega^2 I_c\right) \end{pmatrix} \begin{pmatrix} X \\ \Theta \end{pmatrix} = 0 \tag{9.27}$$

Simplifying Eq. (9.27) with the substitution of given numerical quantities gives the characteristic equation as

$$(73{,}000 - 1460\omega^2)(172{,}942.7 - 2170\omega^2) - (15{,}890)^2 = 0$$

From this we find the two natural frequencies to be

$$\omega_1 = 6.894 \text{ rad/s}, \qquad \omega_2 = 9.065 \text{ rad/s}$$

The amplitude ratios corresponding to the natural frequencies are found from Eq. (9.20) to be

$$\left(\frac{X}{\Theta}\right)_{\omega_1} = -4.401 \text{ m/rad} \qquad \left(\frac{X}{\Theta}\right)_{\omega_2} = 0.338 \text{ m/rad}$$

Example 9.4

For the system shown in Fig. 9.4, let the initial conditions $x_1(0) = x_2(0) = 0$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$. With the use of Laplace transform method, determine the general solution of the system when m_1 is excited by a harmonic force $F_1 \sin \omega t$. To simplify the consideration, assume $m_1 = m_2 = m$.

Solution. From Example 9.1, we can obtain the equations for the motion as

$$m\ddot{x}_1 + 2kx_1 - kx_2 = F_1 \sin \omega t \tag{9.28}$$

$$m\ddot{x}_2 - kx_1 + 2kx_2 = 0 \tag{9.29}$$

The Laplace transform is a powerful tool for solving linear differential equations as was discussed in Section 8.5. Here we illustrate how the method can be applied

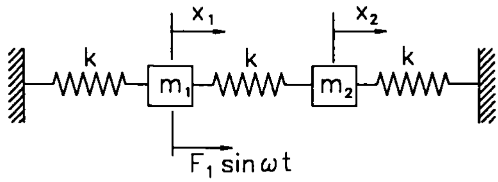

Fig. 9.4 Two-degree-of-freedom system under forced harmonic excitation.

to two equations simultaneously. Taking the Laplace transform of the preceding equations, i.e., multiplying both sides of equations by $e^{-st}dt$ and integrating from zero to infinity, gives

$$ms^2X + 2kX_1 - kX_2 = F_1\frac{\omega}{s^2+\omega^2} \tag{9.30}$$

$$ms^2X_2 - kX_1 + 2kX_2 = 0 \tag{9.31}$$

where X_i is the transformed function of $x_i(t)$ and $(\omega/s^2 + \omega^2) = \mathcal{L}(\sin\omega t)$ obtained from Appendix F. Rewrite the equations in matrix form as

$$\begin{pmatrix} ms^2+2k & -k \\ -k & ms^2+2k \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F_1\dfrac{\omega}{s^2+\omega^2} \\ 0 \end{pmatrix} \tag{9.32}$$

or

$$Z(s)X = F$$

where $Z(s)$ is the coefficient matrix of Eq. (9.32). Premultiplying Eq. (9.32) by the inverse matrix of $Z(s)$ gives

$$X = [Z(s)]^{-1}F$$

or

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = [Z(s)]^{-1}\begin{pmatrix} F_1\dfrac{\omega}{s^2+\omega^2} \\ 0 \end{pmatrix}$$

$$= \frac{\text{adj}[Z(s)]}{|Z(s)|}\begin{pmatrix} F_1\dfrac{\omega}{s^2+\omega^2} \\ 0 \end{pmatrix}$$

$$= \frac{1}{|Z(s)|}\begin{pmatrix} ms^2+2k & k \\ k & ms^2+2k \end{pmatrix}\begin{pmatrix} F_1\dfrac{\omega}{s^2+\omega^2} \\ 0 \end{pmatrix}$$

Carrying out the matrix algebra leads to

$$X_1 = \frac{(ms^2+2k)\omega F_1}{[(ms^2+2k)^2 - k^2](s^2+\omega^2)} \tag{9.33}$$

$$X_2 = \frac{k\omega F_1}{[(ms^2+2k)^2 - k^2](s^2+\omega^2)} \tag{9.34}$$

With the use of the partial fractions expansion given in Appendix E, we can express X_i as

$$X_1 = \frac{\omega F_1}{2(\omega^2 - \omega_1^2)m}\frac{1}{s^2+\omega_1^2} + \frac{\omega F_1}{2(\omega^2 - \omega_2^2)m}\frac{1}{s^2+\omega_2^2} + \frac{\omega_1^2(2\omega_1^2 - \omega^2)F_1}{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)k}\frac{\omega}{s^2+\omega^2} \tag{9.35}$$

$$X_2 = \frac{\omega\omega_1 F_1}{2k(\omega^2 - \omega_1^2)}\frac{\omega_1}{s^2+\omega_1^2} - \frac{\omega\omega_2 F_1}{6k(\omega^2 - \omega_2^2)}\frac{\omega_2}{s^2+\omega_2^2} + \frac{\omega_1^2\omega_2^2 F_1}{3k(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}\frac{\omega}{s^2+\omega^2} \tag{9.36}$$

where $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. Taking inverse Laplace transform, we find

$$x_1(t) = \frac{\omega\omega_1 F_1}{2k(\omega^2 - \omega_1^2)}\sin\omega_1 t + \frac{\omega\omega_2 F_1}{6k(\omega^2 - \omega_2^2)}\sin\omega_2 t + \frac{\omega_1^2(2\omega_1^2 - \omega^2)F_1}{k(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}\sin\omega t \tag{9.37}$$

$$x_2(t) = \frac{\omega\omega_1 F_1}{2k(\omega^2 - \omega_1^2)}\sin\omega_1 t - \frac{\omega\omega_2 F_1}{6k(\omega^2 - \omega_2^2)}\sin\omega_2 t + \frac{\omega_1^2\omega_2^2 F_1}{3k(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}\sin\omega t \tag{9.38}$$

From this result we can conclude that the system will vibrate at the combination of three frequencies. Resonance will take place as ω approaches either ω_1 or ω_2. Note also that the Laplace transform method is very systematic and straightforward.

Example 9.5

Consider a damped system with two degrees of freedom as shown in Fig. 9.5. Find the equations of motion. Determine the natural frequencies and the response of principal modes. Discuss all possible cases for different roots of the characteristic equation.

Solution. From the balance of forces in the free-body diagram, we find

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - c_1\dot{x}_1 - c_2(\dot{x}_1 - \dot{x}_2) \tag{9.39}$$

$$m_2\ddot{x}_2 = k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) \tag{9.40}$$

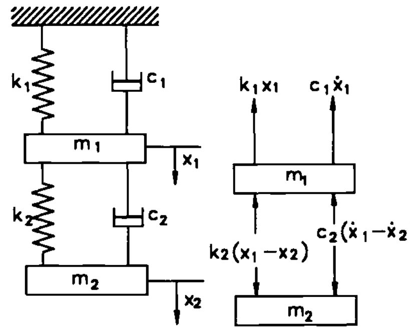

Fig. 9.5 Damped system with two degrees of freedom.

which can be rearranged as

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = 0$$
$$-c_2\dot{x}_1 - k_2x_1 + m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 = 0$$

Assume that the solutions are in the form of

$$x_1 = X_i e^{st}$$

Then we have

$$\begin{aligned} &[m_1s^2 + (c_1 + c_2)s + (k_1 + k_2)]X_1 - (c_2s + k_2)X_2 = 0 \\ &-(c_2s + k_2)X_1 + (m_2s^2 + c_2s + k_2)X_2 = 0 \end{aligned} \tag{9.41}$$

Because X_1 and X_2 cannot be zero, the determinant of the coefficients must be zero. Expanding the determinant gives

$$\begin{aligned} &m_1m_2s^4 + [m_1c_2 + m_2(c_1 + c_2)]s^3 + [m_1k_2 + m_2(k_1 + k_2) \\ &+ c_1c_2]s^2 + (k_1c_2 + k_2c_1)s + k_1k_2 = 0 \end{aligned} \tag{9.42}$$

From this equation, s is expected to have four roots. When these roots are substituted into Eq. (9.41), they will give four relationships between X_1 and X_2. Note that because all the physical constants m_i, k_i, and c_i are positive and all the signs are plus, there is no possibility of a positive root. Thus the following possibilities exist for the four roots: 1) all four roots are complex numbers that will be two pairs of complex conjugates; 2) all four roots are real and negative; and 3) two roots are real and negative, and the other two complex conjugates.

Now let us examine these three possible cases. For the two pairs of complex conjugates, i.e.,

$$\begin{aligned} s_1 &= -p_1 + iq_1, \qquad s_2 = -p_1 - iq_1 \\ s_3 &= -p_2 + iq_2, \qquad s_4 = -p_2 - iq_2 \end{aligned} \tag{9.43}$$

where p_1, p_2, q_1, and q_2 are real and positive. The first two roots will give the following solutions:

$$\begin{aligned} x_1 &= X_{11}\exp[(-p_1 + iq_1)t] + X_{12}\exp[(-p_1 - iq_1)t] \\ &= e^{-p_1 t}(X_{11}e^{iq_1 t} + X_{12}e^{-iq_1 t}) = A_{11}e^{-p_1 t}\sin(q_1 t + \phi_{11}) \end{aligned} \tag{9.44}$$

and

$$\begin{aligned} x_2 &= X_{21}\exp[(-p_1 + iq_1)t] + X_{22}\exp[(-p_1 - iq_1)t] \\ &= A_{21}e^{-p_1 t}\sin(q_1 t + \phi_{21}) \end{aligned} \tag{9.45}$$

These two solutions represent oscillatory motion with the magnitudes decaying exponentially. In a similar way, for roots s_3 and s_4, we have another two solutions. Combining all four roots, the general solutions are then

$$x_1 = A_{11}e^{-p_1 t}\sin(q_1 t + \phi_{11}) + A_{12}e^{-p_2 t}\sin(q_2 t + \phi_{12}) \tag{9.46}$$

$$x_2 = A_{21}e^{-p_1 t}\sin(q_1 t + \phi_{21}) + A_{22}e^{-p_2 t}\sin(q_2 t + \phi_{22}) \tag{9.47}$$

where $A_{11}, A_{12}, A_{21}, A_{22}, \phi_{11}, \phi_{12}, \phi_{21}$, and ϕ_{22} are to be determined. With the use of Eq. (9.41), for each root, four relationships are established. Another four relationships can be found by the four initial conditions $x_1(0), x_2(0), \dot{x}_1(0)$, and $\dot{x}_2(0)$. Therefore all the constants will be determined.

For the second case, four roots are real and negative, then the motion is not oscillatory; the displacements of masses are decaying exponentially. This case is similar to the overdamped case discussed in Section 8.4.

Finally, for the third case, two roots are real and negative, and the other two are a pair of complex conjugates. The general solutions are then the combination of the terms, as in Eq. (9.44), and the other terms of exponential functions:

$$x_i = A_i e^{-pt}\sin(qt + \phi_i) + c_i e^{-s_3 t} + d_i e^{-s_4 t}$$

The constants are determined through the same procedures as discussed for the first case.

9.2 Matrix Formulation for Systems with Multiple Degrees of Freedom

There are usually n ordinary differential equations for describing a system of n degrees of freedom. Solving these equations is straightforward but cumbersome and time-consuming if n is large. Fortunately, matrix methods are ideal for this purpose, and many matrix operations can be carried out by digital computers. In this section we will discuss the matrix techniques for various properties of vibrating systems.

Free Vibration of Undamped Systems

The equations of motion for an n degrees-of-freedom system expressed in matrix form are simplified to

$$M\ddot{X} + KX = 0 \tag{9.48}$$

where the mass matrix is

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix}$$

the stiffness matrix is

$$K = \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ \vdots & \vdots & & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{pmatrix}$$

and the displacement vector (a column matrix) is

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Note that M and K are square symmetric matrices. Premultiplying Eq. (9.48) by M^{-1}, we find

$$I\ddot{X} + M^{-1}KX = 0$$

or

$$I\ddot{X} + AX = 0 \tag{9.49}$$

where $A = M^{-1}k$ is called the system matrix or the dynamic matrix because the dynamic properties of the system are defined by this matrix. By assuming the solution of the equation in the form of

$$X = Ce^{i\omega t}$$

we have

$$\ddot{X} = -\omega^2 X$$

or

$$\ddot{X} = -\lambda X$$

where $\lambda = \omega^2$. Then Eq. (9.49) becomes

$$(A - \lambda I)X = 0$$

Because X is not zero, the determinant of the coefficients must be zero, i.e.,

$$|A - \lambda I| = 0 \tag{9.50}$$

This is the characteristic equation of the system. From this equation we can find n roots of λ_i, which are called eigenvalues. By substituting λ_i into Eq. (9.51), we can obtain the corresponding mode shape x_i, which is called an eigenvector. Note that, as λ_i is substituted into Eq. (9.49), there are most likely only $(n-1)$ independent equations, but there are various nx_i to be determined. One x_i can be chosen arbitrarily. It is convenient to add one condition as

$$\sum_i x_i^2 = 1 \tag{9.51}$$

In this way, x_i may be considered direction cosine throughout for two- and three-degree-of-freedom systems. For $n > 3$, the additional condition (9.51) is still valid to replace the condition that one x_i is arbitrarily chosen. Details will be shown in the examples.

Eigenvalue and eigenvector properties: different eigenvalues $\lambda_i \neq \lambda_j$. For the ith mode, we have

$$AX_i = \lambda_i X_i \tag{9.52}$$

If the transposed equation (9.52) is postmultiplied by X_j, then it becomes

$$(AX_i)^T X_j = \lambda_i X_i^T X_j$$

or

$$X_i^T AX_j = \lambda_i X_i^T X_j \tag{9.53}$$

On the other hand, for the jth mode, the equation is

$$AX_j = \lambda_j X_j$$

Premultiplying the preceding equation by X_i^T gives

$$X_i^T AX_j = \lambda_j X_i^T X_j \tag{9.54}$$

When Eq. (9.53) is subtracted by Eq. (9.54), we find

$$(\lambda_i - \lambda_j) X_i^T X_j = 0$$

Therefore, X_i and X_j are orthogonal.

In addition, consider the equation for the ith mode

$$KX_i = \lambda_i MX_i$$

Premultiplying the equation by X_j^T gives

$$X_j^T KX_i = \lambda_i X_j^T MX_i \tag{9.55}$$

Next, starting with the equation for the jth mode and premultiplying by X_i^T, we obtain

$$X_i^T KX_j = \lambda_j X_i^T MX_j$$

Taking the transpose of the preceding equation leads to

$$X_j^T K X_i = \lambda_j X_j^T M X_i \tag{9.56}$$

because M and K are symmetric matrices. Thus, subtracting Eq. (9.56) from Eq. (9.55) gives

$$0 = (\lambda_i - \lambda_j) X_j^T M X_i$$

For $\lambda_i \neq \lambda_j$, the preceding equation requires

$$X_j^T M X_i = 0 \tag{9.57}$$

It is also evident from Eq. (9.55) that

$$X_j^T K X_i = 0 \qquad \text{for } i \neq j \tag{9.58}$$

On the other hand, as $i = j$, we write

$$X_i^T M X_i = M_i$$

and

$$X_i^T K X_i = K_i$$

M_i and K_i are called generalized mass and generalized stiffness, respectively.

Eigenvalue and eigenvector properties: repeated eigenvalues $\lambda_i = \lambda_j$. Suppose that there are three roots from the characteristic equation, with $\lambda_1 = \lambda_2 = \lambda_0$ and $\lambda_3 \neq \lambda_0$. Then we have

$$\begin{aligned} Ax_1 &= \lambda_0 x_1 \\ Ax_2 &= \lambda_0 x_2 \\ Ax_3 &= \lambda_3 x_3 \end{aligned} \tag{9.59}$$

Multiplying the second equation by any constant b and adding it to the first gives

$$A(x_1 + bx_2) = \lambda_0(x_1 + bx_2)$$

Thus a new eigenvector $x_{12} = x_1 + bx_2$ also satisfies the equation; hence, no unique eigenvector exists for λ_0. However, based on orthogonal properties of eigenvectors, we can choose x_1 to be perpendicular to x_3 and x_2 perpendicular to x_1 and x_3. Details will be shown in the example.

Principal or normal coordinates. With the properties of eigenvalues and eigenvectors already discussed, we can transform the equation of motion from Eq. (9.48)

$$M\ddot{X} + KX = 0 \tag{9.48}$$

to

$$\ddot{Y}_i + \omega_i^2 Y_i = 0$$

by the transformation of

$$X = PY \tag{9.60}$$

where P is called the modal matrix and is formed by eigenvectors. For a three-degree-of-freedom system

$$P = \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_1 \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_2 \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_3 \right) = (X_1 \quad X_2 \quad X_3)$$

where X_1, X_2, X_3 are eigenvectors. With the transformation by Eq. (9.57), Eq. (9.56) becomes

$$MP\ddot{Y} + KPY = 0$$

Premultiplying the preceding equation by P^T gives

$$P^T MP\ddot{Y} + P^T KPY = 0 \tag{9.61}$$

Looking into details, we find

$$P^T MP = (X_1 \quad X_2 \quad X_3)^T (M)(X_1 \quad X_2 \quad X_3)$$

$$= \begin{pmatrix} X_1^T MX_1 & X_1^T MX_2 & X_1^T MX_3 \\ X_2^T MX_1 & X_2^T MX_2 & X_2^T MX_3 \\ X_3^T MX_1 & X_3^T MX_2 & X_3^T MX_3 \end{pmatrix}$$

$$= \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}$$

where $M_i = X_i^T MX_i$ and Eq. (9.57) has been used for the zero terms. Similarly

$$P^T KP = \begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix}$$

Therefore, Eq. (9.61) becomes

$$M_i\ddot{Y}_i + K_iY_i = 0 \qquad i = 1, 2, 3$$

which can be solved in a manner similar to that of the single-degree-of-freedom system. Once Y_i is found, the solution of the original equation can be obtained simply by applying the transformation equation

$$X(t) = PY$$

Example 9.6

Consider the system shown in Fig. 9.1 with $m_1 = m_2 = m$. Find the steady-state solution with the use of principal coordinates.

Solution. The equation of motion in matrix form is

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{9.62}$$

and the eigenvalues and eigenvectors are found

$$\lambda_1 = \omega_1^2 = k/m, \qquad \lambda_2 = \omega_2^2 = 3(k/m)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\lambda_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\lambda_2} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Hence the modal matrix P is

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The transformation is

$$X = PY$$

Equation (9.62) then becomes

$$MP\ddot{Y} + KPY = 0$$

Premultiplying the preceding equation by P^T, we find

$$P^T MP\ddot{Y} + P^T KPY = 0$$

or

$$m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} + k \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

This equation can be further simplified to

$$\ddot{y}_i + \omega_i^2 y_i = 0 \quad i = 1, 2 \tag{9.63}$$

The general solution of Eq. (9.63) is

$$y_i(t) = y_i(0) \cos \omega_i t + (1/\omega_i) \dot{y}_i(0) \sin \omega_i t$$

The initial conditions for the principal coordinates can be found from the transformation equation as follows:

$$Y = P^T X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$y_1(0) = \frac{1}{\sqrt{2}}[x_1(0) + x_2(0)]$$

$$y_2(0) = \frac{1}{\sqrt{2}}[x_2(0) - x_1(0)]$$

Similar relationships can be found for $\dot{y}_i(0)$. Therefore

$$y_1(t) = \frac{1}{\sqrt{2}}[x_1(0) + x_2(0)] \cos \omega_1 t + \frac{1}{\sqrt{2}\omega_1}[\dot{x}_1(0) + \dot{x}_2(0)] \sin \omega_1 t$$

$$y_2(t) = \frac{1}{\sqrt{2}}[x_2(0) - x_1(0)] \cos \omega_2 t + \frac{1}{\sqrt{2}\omega_2}[\dot{x}_2(0) + \dot{x}_1(0)] \sin \omega_2 t$$

To find the solution for x_1, x_2, we substitute $y_i(t)$ into the transformation equation

$$X = PY$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1 - y_2 \\ y_1 + y_2 \end{pmatrix}$$

Therefore

$$\begin{aligned} x_1 &= \frac{1}{2}[x_1(0) + x_2(0)] \cos \omega_1 t + \frac{1}{2\omega_1}[\dot{x}_1(0) + \dot{x}_2(0)] \sin \omega_1 t \\ &+ \frac{1}{2}[x_1(0) - x_2(0)] \cos \omega_2 t + \frac{1}{2\omega_2}[\dot{x}_1(0) - \dot{x}_2(0)] \sin \omega_2 t \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{1}{2}[x_1(0) + x_2(0)] \cos \omega_1 t + \frac{1}{2\omega_1}[\dot{x}_1(0) + \dot{x}_2(0)] \sin \omega_1 t \\ &- \frac{1}{2}[x_1(0) - x_2(0)] \cos \omega_2 t - \frac{1}{2\omega_2}[\dot{x}_1(0) - \dot{x}_2(0)] \sin \omega_2 t \end{aligned}$$

The preceding results agree completely with Eqs. (9.16) and (9.17).

Example 9.7

To illustrate a case of repeated roots in a characteristic equation, let us consider a particular system with the equation of motion as

$$\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{pmatrix} 0 & -k & k \\ -k & 0 & k \\ k & k & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

When this equation is premultiplied by M^{-1}, the equation becomes

$$\ddot{X} + M^{-1}KX = 0$$

or

$$\ddot{X} + AX = 0 \tag{9.64}$$

where

$$A = M^{-1}K = \frac{k}{m}\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{9.65}$$

1) Find the eigenvalues and eigenvectors, and 2) find the modal matrix P and carry out the product $P^T AP$.

Solution. 1) The steady-state solution of Eq. (9.64) may be assumed as

$$X = Ce^{i\omega t}$$

$$\ddot{X} = -C\omega^2 e^{i\omega t} = -C\lambda e^{i\omega t}$$

Substituting the preceding expression into Eq. (9.64) leads to

$$(A - \lambda I)X = 0 \tag{9.66}$$

Hence the characteristic equation is

$$|A - \lambda I| = 0$$

or

$$\lambda^3 - 3\left(\frac{k}{m}\right)^2 \lambda + 2\left(\frac{k}{m}\right)^3 = 0$$

$$\left(\lambda - \frac{k}{m}\right)^2 \left(\lambda + 2\frac{k}{m}\right) = 0$$

which gives the eigenvalues

$$\lambda_1 = \lambda_2 = k/m \qquad \text{and} \qquad \lambda_3 = -2k/m$$

To find the eigenvector corresponding to $\lambda_3 = -2k/m$, we substitute λ_3 into Eq. (9.66) and obtain

$$\frac{k}{m}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

In explicit form, we have

$$2x_1 - x_2 + x_3 = 0$$

$$-x_1 + 2x_2 + x_3 = 0$$

$$x_1 + x_2 + 2x_3 = 0$$

Note that there are only two independent equations in the preceding three equations. For example, the second equation can be obtained by the subtraction of the first equation from the third equation. Fortunately we can impose

$$x_1^2 + x_2^2 + x_3^2 = 1$$

Then we find the eigenvector

$$X_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \tag{9.67}$$

For $\lambda_1 = \lambda_2 = k/m$, the equations become

$$-x_1 - x_2 + x_3 = 0$$

$$-x_1 - x_2 + x_3 = 0$$

$$x_1 + x_2 - x_3 = 0$$

There is only one independent equation in the preceding equations. Taking $x_1 = x_3 - x_2$ leads to

$$X_1 = \begin{pmatrix} x_3 - x_2 \\ x_2 \\ x_3 \end{pmatrix}$$

To satisfy the orthogonality condition $X_1^T X_3 = 0$, we have $x_2 = x_3$ and also to satisfy $x_1^2 + x_2^2 + x_3^2 = 1$. Hence

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{9.68}$$

The second eigenvector for $\lambda = k/m$ then can be constructed from

$$X_2 = X_1 \times X_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{9.69}$$

2) The modal matrix P is obtained by collecting the eigenvectors from Eqs. (9.69–9.71)

$$P = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{9.70}$$

The product of $P^T AP$ is found to be

$$P^T AP = \frac{k}{m} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Note that the result is a diagonal matrix of the eigenvalues.

Forced Vibration of Undamped Systems

The vibration of systems with multiple degrees of freedom activated by harmonic forcing functions can be treated quite simply as an extension of our matrix methods. Consider a system with three degrees of freedom and with forces $F_1(t) = q_1 e^{i\omega t}$, $F_2(t) = q_2 e^{i\omega t}$, and $F_3(t) = q_3 e^{i\omega t}$ being applied in the directions of x_1, x_2, and x_3, respectively. The equation of motion can be written as

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} e^{i\omega t} \tag{9.71}$$

By assuming the steady-state solution as

$$x_i = X_i e^{i\omega t} \tag{9.72}$$

we have

$$-M\omega^2 X + KX = Q$$

where M is the mass matrix, K is the stiffness matrix, and X is the column matrix of X_i. Further simplifying the equation gives

$$AX = Q$$

where $A = K - M\omega^2$. By premultiplying the equation by A^{-1}, we find immediately,

$$X = A^{-1}Q \tag{9.73}$$

Forced Vibration of Viscously Damped Systems

The differential equations of motion for a damped system having n degrees of freedom can be written in matrix form as

$$M\ddot{X} + C\dot{X} + KX = F \tag{9.74}$$

where M, C, K are $n \times n$ symmetric matrices and X, F are $n \times 1$ column matrices. To find the homogeneous solution, we set $F = 0$ and assume solutions of the form

$$x_i(t) = X_i e^{st} \quad i = 1, 2, \ldots, n$$

Substitution of the assumed solutions yields the matrix equation

$$s^2 MX + sCX + KX = 0$$

or

$$(s^2 M + sC + K)X = 0$$

To find the eigenvalues, we set the determinant of the coefficients to zero:

$$|s^2 M + sC + K| = 0$$

This is the characteristic equation. From this equation, we expect to find n roots or n eigenvalues. Then we can find n corresponding eigenvectors. However, the n roots are usually n pairs of complex conjugates and n eigenvectors are also in complex form. Therefore, very often we specify that the solution is the real part of the assumed solution, i.e.,

$$x_i(t) = \mathrm{Re}\left[X_i e^{S_i t}\right]$$

To find the particular solution of Eq. (9.74), we consider the following two special cases of damping systems.

Light damping. From the homogeneous undamped equation

$$M\ddot{X} + KX = 0$$

we obtain the eigenvalues and eigenvectors. From this we can transform X to principal coordinates Y by

$$X = PY$$

Substituting this transformation into Eq. (9.74) and premultiplying the equation by P^T, we have

$$P^T M P \ddot{Y} + P^T C P \dot{Y} + P^T K P Y = P^T F \tag{9.75}$$

It has been shown previously that $P^T M P$ and $P^T K P$ are diagonal matrices. In general, $P^T C P$ results in a nondiagonal matrix. A frequently used approach for approximating the response of a system with light damping is to ignore all off-diagonal terms of the transformed damping matrix, then Eq. (9.75) becomes n uncoupled equations. Each can be solved by the methods used for a single-degree-of-freedom system already discussed.

Proportional damping. If C is proportional to M and K

$$C = \alpha M + \beta K \tag{9.76}$$

where α and β are constants, then

$$P^T C P = \alpha P^T M P + \beta P^T K P$$

Thus Eq. (9.75) becomes uncoupled. Each principal coordinate will have the equation of motion of the form

$$\ddot{Y}_1 + \left(\alpha + \beta\omega_i^2\right)\dot{Y}_i + \omega_i^2 Y_i = f_i(t) \tag{9.77}$$

which can be solved by the methods discussed in Chapter 8.

9.3 Lumped Parameter Systems with Transfer Matrices

Many vibrational systems can be modeled as systems with lumped parameters. The method of transfer matrices is introduced here. This is a powerful tool for solving lumped parameter systems. To establish the method, we first apply the method for mass-spring systems. Then we will apply it to torsional systems and flexural beam systems. The method requires the knowledge of matrix operation, which has been reviewed in previous sections.

State Vectors and Transfer Matrices

To apply the method of lumped parameters to a vibration system, we divide the system into a number of appropriate sections. For each section, physical quantities are classified into two kinds of variables. One kind is known as the force, which includes force, torque, shear, and bending moment, and the other as the displacement, which includes linear displacement and angular displacement.

Now, we define two terms, state vector and transfer matrix, that are used in the method of lumped parameters. A state vector is a column matrix that has all of the components of the forces and displacement at a point i. The transfer matrix relates the state vectors from one location to another along the system.

Let us consider a mass and a spring as shown in Fig. 9.6a. We can formulate two equations: one is for the force, and the other for the displacement. For the force, we have

$$\sum f_i = m\ddot{x}_i$$

$$f_i - f_{i-1} = m\ddot{x}_i$$

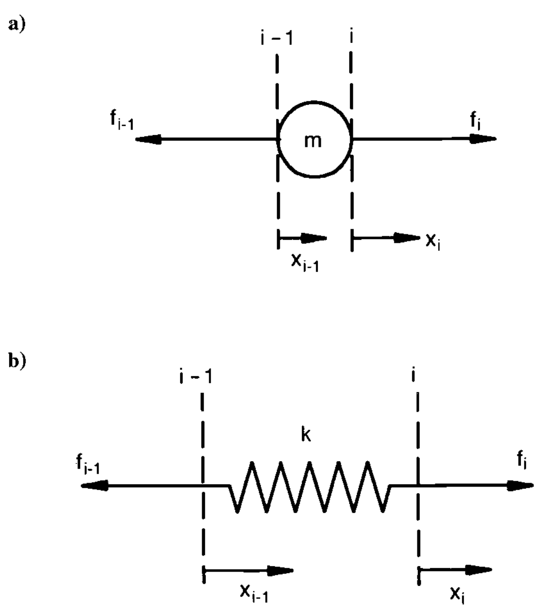

Fig. 9.6 State variables for a mass-spring system.

For a harmonic motion, the solution is assumed as $x_i = X_i e^{i\omega t}$. The forces applied must be in the same form $f = F e^{i\omega t}$. Hence we have

$$F_i = F_{i-1} - m\omega^2 X_i$$

For the displacement, we have

$$x_i = x_{i-1}$$

or

$$X_i = X_{i-1}$$

In matrix form

$$\begin{pmatrix} X_i \\ F_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{pmatrix} \begin{pmatrix} X_{i-1} \\ F_{i-1} \end{pmatrix} \tag{9.78}$$

Next, let us consider the state variables around the spring as shown in Fig. 9.6b; we have

$$\sum F = 0$$

$$f_i = f_{i-1} = k(x_i - x_{i-1})$$

or

$$F_i = F_{i-1} = k(X_i - X_{i-1})$$

Again, in matrix form

$$\begin{pmatrix} X_i \\ F_i \end{pmatrix} = \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_{i-1} \\ F_{i-1} \end{pmatrix} \tag{9.79}$$

In Eqs. (9.78) and (9.79), the state vector is

$$Z_i = \begin{pmatrix} X_i \\ F_i \end{pmatrix}$$

The transfer matrices in Eq. (9.78) and (9.79) are, respectively,

$$A = \begin{pmatrix} 1 & 0 \\ -m\omega^2 & -1 \end{pmatrix} \tag{9.80}$$

$$B = \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix} \tag{9.81}$$

Therefore, for a system consisting of a spring and a mass as shown in Fig. 9.7, we can write the equations as

$$Z_i = B Z_{i-1}$$

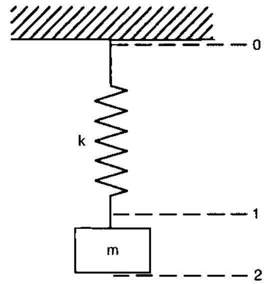

Fig. 9.7 Mass-spring system.

for the relationship across the spring and

$$Z_{i+1} = AZ_i$$

for that across the mass. The combined equation then is

$$Z_{i+1} = ABZ_{i-1} \tag{9.82}$$

Example 9.8

For a mass-spring system as shown in Fig. 9.7, determine the natural frequency of the system using state vectors and transfer matrices.

Solution. Applying the formulation given in the section, we have the transfer matrices as

$$B_{1-0} = \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix}$$

and

$$A_{2-1} = \begin{pmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{pmatrix}$$

The equation of motion is

$$Z_2 = A_{2-1}B_{1-0}Z_0$$

In detail, we have

$$\begin{aligned} \begin{pmatrix} X_2 \\ F_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_0 \\ F_0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1/k \\ -m\omega^2 & (1 - m\omega^2/k) \end{pmatrix} \begin{pmatrix} X_0 \\ F_0 \end{pmatrix} \end{aligned} \tag{9.83}$$

Two different conditions for the vibrations may be illustrated as follows: 1) free vibration and 2) forced vibration.

1) The conditions for free vibration are $F_2 = 0$ and $X_0 = 0$. Then Eq. (9.83) leads to two equations

$$X_2 = F_0/k$$

and

$$0 = (1 - m\omega^2/k)F_0$$

which give the displacement of the mass in terms of the force in the spring and the natural frequency

$$\omega = \sqrt{k/m}$$

2) For a harmonically forced vibration $f_2 = Fe^{i\omega t}$, the magnitude of the force is F. Hence the conditions can be written as $F_2 = F$ and $X_0 = 0$. Again Eq. (9.83) gives

$$X_2 = F_0/k$$

and

$$F = (1 - m\omega^2/k)F_0$$

Rearranging leads to

$$X_2 = \frac{F}{k(1 - m\omega^2/k)} \tag{9.84}$$

which is the familiar result.

Transfer Matrices for Torsional Systems

Consider a disk and a bar as shown in Fig. 9.8a. As we have done in the last section, we formulate one equation for force and one for displacement. From the balance of torque, we have

$$t_i - t_{i-1} = J\ddot{\theta}_i \qquad \theta_i = \theta_{i-1}$$

Under the harmonic vibration, we express the state vector as

$$z_i = Z_i e^{i\omega t}$$

where the capital letter represents the magnitude. Therefore, the equations written in matrix form become

$$\begin{pmatrix} \Theta_i \\ T_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -J\omega^2 & 1 \end{pmatrix} \begin{pmatrix} \Theta_{i-1} \\ T_{i-1} \end{pmatrix} \tag{9.85}$$

Next consider Fig. 9.8b, and assume that the bar is an elastic torsional spring. Then we find

$$t_i = t_{i-1} = k(\theta_i - \theta_{i-1})$$

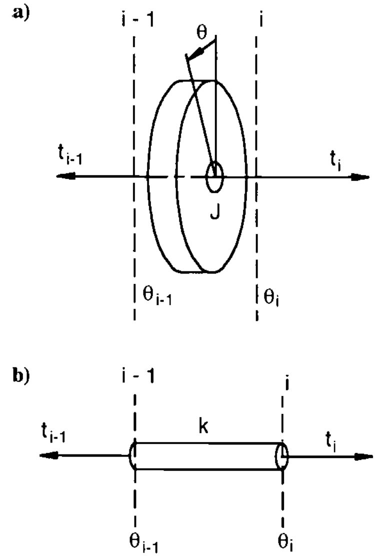

Fig. 9.8 State variables for a torsional system.

which can be written in matrix form as

$$\begin{pmatrix} \Theta_i \\ T_i \end{pmatrix} = \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Theta_{i-1} \\ T_{i-1} \end{pmatrix} \tag{9.86}$$

Similar to the case of mass-spring system, if from $i - 1$ to i is a torsional spring we can have the equation

$$Z_i = BZ_{i-1}$$

where

$$B = \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix} \tag{9.87}$$

For the case where from i to $i + 1$ is a disk, the equation is

$$Z_{i+1} = AZ_i \tag{9.88}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ -J\omega^2 & 1 \end{pmatrix}$$

Then the combined equation from $i - 1$ to $i + 1$ is

$$Z_{i+1} = ABZ_{i-1} \tag{9.89}$$

Note that this formulation can be applied to many successive stations of the torsional system. Then the final equation is in the form of

$$Z_n = (AB)_n(AB)_{n-1}, \ldots, (AB)_1 Z_0 \tag{9.90}$$

Example 9.9

For the torsional system shown in Fig. 9.9, employ transfer matrix to find the relationship from station 0 to 3. Determine the natural frequencies for the principal modes.

Solution. To simplify the consideration while not losing generality, let us consider that the boundary conditions are $\theta_0 = 0$, $T_0 = 1$, $\theta_3 = \theta$, and $T_3 = 0$ where θ is arbitrary. The equation relating station 0 to 1 can be written as

$$\begin{pmatrix}\theta_1\\T_1\end{pmatrix} = \begin{pmatrix}1 & 0\\ -3J\omega^2 & 1\end{pmatrix}\begin{pmatrix}1 & 1/(2k)\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$
$$= \begin{pmatrix}1 & 1/(2k)\\ -3J\omega^2 & 1-3J\omega^2/(2k)\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

In a similar manner, to relate stations 1 to 2 and 2 to 3, we have

$$\begin{pmatrix}\theta_3\\T_3\end{pmatrix} = \begin{pmatrix}1 & 1/k\\ -J\omega^2 & (1-3J\omega^2/k)\end{pmatrix}\begin{pmatrix}1 & 1/(1.5k)\\ -2J\omega^2 & (1-2J\omega^2/1.5k)\end{pmatrix}$$
$$\times\begin{pmatrix}1 & 1/(2k)\\ -3J\omega^2 & (1-3J\omega^2/2k)\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

Carrying out the product of the matrices, we find

$$T_3 = \frac{1}{2k}\left[-J\omega^2 - 2J\omega^2\left(1-\frac{J\omega^2}{k}\right)\right]$$
$$+\left[-\frac{J\omega^2}{1.5k} + \left(1-\frac{J\omega^2}{k}\right)\left(1-\frac{2J\omega^2}{1.5k}\right)\right]\left(1-\frac{3J\omega^2}{2k}\right) \tag{9.91}$$

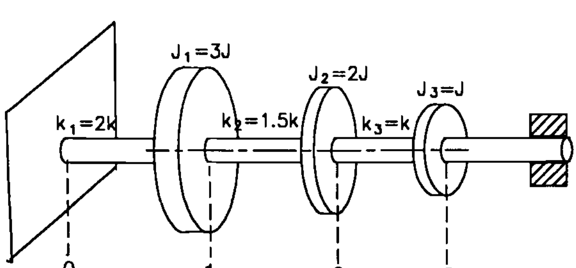

Fig. 9.9 States for the torsional system.

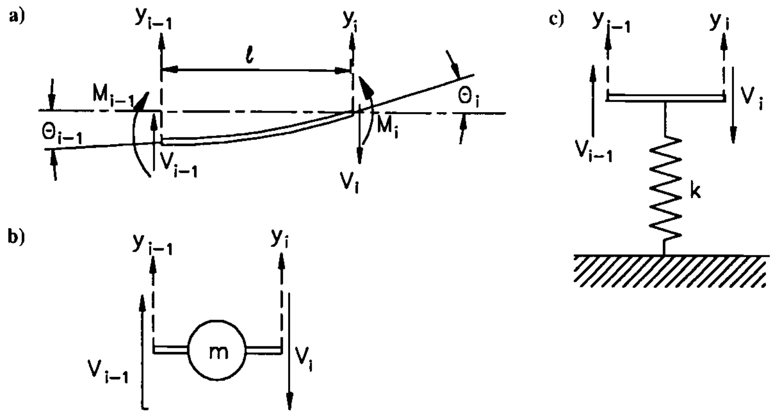

Fig. 9.10 Elements of lumped system for a vibrating beam.

To simplify the equation, let $\omega^2 = \lambda(k/J)$ and set $T_3 = 0$, we obtain

$$2\lambda^3 - \frac{41}{6}\lambda^2 + 6\lambda - 1 = 0$$

and find the three roots as

$$\lambda_1 = 0.2168, \qquad \lambda_1 = 1.0964, \qquad \lambda_3 = 2.1034$$

Therefore the natural frequencies of the principal modes are

$$\omega_1 = 0.4656\sqrt{\frac{k}{J}}, \qquad \omega_2 = 1.0471\sqrt{\frac{k}{J}}, \qquad \omega_3 = 1.4503\sqrt{\frac{k}{J}}$$

Transfer Matrices for Vibrating Beams

A beam is a continuous solid, but it also can be modeled as lumped masses connected by massless beam sections. The lateral vibration of the beam can be solved successfully by using state vectors and transfer matrices. The method originally developed by N. O. Myklestad and adapted by many textbooks* is discussed in this section.

To formulate the governing equations, a portion of the beam is broken into three elements as shown in Fig. 9.10. They are the massless beam section, the mass section, and the load section.

The massless beam section as shown in Fig. 9.10a. There are four equations to relate state vector at station $i - 1$ to that at station i. From $\sum F = 0$, we have

$$V_i = V_{i-1} \tag{9.92}$$

*Thomson, W. T., *Theory of Vibration with Applications*, 3rd ed., Prentice-Hall, Englewood Cliffs, NJ, 1988.

From $\sum M = 0$, we obtain

$$M_i = M_{i-1} + V_{i-1}\ell \tag{9.93}$$

From the relationship between the change of slope and the moment applied in the beam, we have

$$\begin{aligned} \theta_i - \theta_{i-1} &= \frac{1}{EI}\int_0^\ell M(x)\,\mathrm{d}x = \frac{1}{EI}\left(M_{i-1} + \frac{1}{2}V_{i-1}\ell\right)\ell \\ \theta_i &= \theta_{i-1} + \frac{M_{i-1}\ell}{EI} + \frac{V_{i-1}\ell^2}{2EI} \end{aligned} \tag{9.94}$$

The deflection is found similarly

$$Y_i - Y_{i-1} = \int_0^\ell \theta(x)\,\mathrm{d}x = \theta_{i-1}\ell + \frac{M_{i-1}\ell^2}{2EI} + \frac{V_{i-1}\ell^3}{6EI}$$

$$Y_i = Y_{i-1} + \theta_{i-1}\ell + \frac{M_{i-1}\ell^2}{2EI} + \frac{V_{i-1}\ell^3}{6EI} \tag{9.95}$$

Combining Eqs. (9.92–9.95) leads to

$$\begin{pmatrix} Y_i \\ \theta_i \\ M_i \\ V_i \end{pmatrix} = \begin{pmatrix} 1 & \ell & \ell^2/2EI & \ell^3/6EI \\ 0 & 1 & \ell/EI & \ell^2/2EI \\ 0 & 0 & 1 & \ell \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{i-1} \\ \theta_{i-1} \\ M_{i-1} \\ V_{i-1} \end{pmatrix} \tag{9.96}$$

Note that there are four terms in the state vector. Once the transfer matrix is determined for one section, it can be used for all sections of the same length and same flexural rigidity.

The mass section as shown in Fig. 9.10b. From the equation of motion

$$\sum F = m\ddot{y}$$

we have

$$V_{i-1} - V_i = m\ddot{y}$$

By assuming the harmonic vibration and keeping V_i for the magnitudes of harmonic shear forces

$$V_{i-1} - V_i = -mY_i\omega^2$$

For a rigid mass

$$Y_i = Y_{i-1}$$

$$V_i = V_{i-1} + m\omega^2 Y_{i-1}$$

$$\theta_i = \theta_{i-1}$$

$$M_i = M_{i-1}$$

Combining the preceding equations into one transfer matrix equation, we have

$$\begin{pmatrix} Y_i \\ \theta_i \\ M_i \\ V_i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m\omega^2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{i-1} \\ \theta_{i-1} \\ M_{i-1} \\ V_{i-1} \end{pmatrix} \tag{9.97}$$

The load section as shown in Fig. 9.10c. The balance of forces gives

$$V_{i-1} = V_i + kY_i$$

Because Y, θ, and M are not changed from station $i - 1$ to i, the transfer across a spring is simply

$$\begin{pmatrix} Y_i \\ \theta_i \\ M_i \\ V_i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -k & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{i-1} \\ \theta_{i-1} \\ M_{i-1} \\ V_{i-1} \end{pmatrix} \tag{9.98}$$

Now we have used three elements to model a vibrating beam and obtained three sets of matrix equations. Note that the dimensions of each term in the transfer matrix are different. It will be more convenient if all the terms are written in dimensionless form, especially if we use a computer to carry out the matrix operations. Let us define dimensionless variables as follows:

$$Y_i^* = \frac{Y_i}{\ell}, \qquad M_i^* = \frac{M_i \ell}{EI}, \qquad V_i^* = \frac{V_i \ell^2}{EI}$$

$$m^* = \frac{m\omega^2 \ell^3}{EI}, \qquad k^* = \frac{k\ell^3}{EI}$$

Then Eqs. (9.96–9.98) become respectively,

$$\begin{pmatrix} Y_i^* \\ \theta_i \\ M_i^* \\ V_i^* \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{i-1}^* \\ \theta_{i-1} \\ M_{i-1}^* \\ V_{i-1}^* \end{pmatrix} \tag{9.99}$$

$$\begin{pmatrix} Y_i^* \\ \theta_i \\ M_i^* \\ V_i^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m^* & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{i-1}^* \\ \theta_{i-1} \\ M_{i-1}^* \\ V_{i-1}^* \end{pmatrix} \tag{9.100}$$

$$\begin{pmatrix} Y_i^* \\ \theta_i \\ M_i^* \\ V_i^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -k^* & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{i-1}^* \\ \theta_{i-1} \\ M_{i-1}^* \\ V_{i-1}^* \end{pmatrix} \tag{9.101}$$

Example 9.10

For the uniform cantilever beam shown in Fig. 9.11, find the transfer matrices and determine the natural frequencies and corresponding principal modes of vibration.

Solution. The boundary conditions are $Y_0^* = 0, \theta_0 = 0$, and $M_3^* = 0$, $V_3^* = 0$. The equation relating station 0 to 1 is

$$\begin{pmatrix} Y_1^* \\ \theta_1 \\ M_1^* \\ V_1^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m^* & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ M_0^* \\ V_0^* \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ m^* & m^* & \frac{1}{2}m^* & \left(1 + \frac{1}{6}m^*\right) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ m_0^* \\ V_0^* \end{pmatrix} \tag{9.102}$$

Similarly, for the relationship from station 0 to 2, the equation is

$$\begin{pmatrix} Y_2^* \\ \theta_2 \\ M_2^* \\ V_2^* \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ m^* & m^* & \frac{1}{2}m^* & \left(1 + \frac{1}{6}m^*\right) \end{pmatrix} \begin{pmatrix} Y_1^* \\ \theta_1 \\ M_1^* \\ V_1^* \end{pmatrix} \tag{9.103}$$
$$= A_1 Z_1 = A_1 A_1 Z_0 = A_2 Z_0$$

where Z_i is the state vector at station i and A_i is the transfer matrix.

$$A_2 = A_1 A_1 =$$
$$\begin{pmatrix} (1 + m^*/6) & (2 + m^*/6) & (2 + m^*/12) & (4/3 + m^*/36) \\ m^*/2 & (1 + m^*/2) & (2 + m^*/4) & (2 + m^*/12) \\ m^* & m^* & (1 + m^*/2) & (2 + m^*/6) \\ m^*(2 + m^*/6) & m^*(3 + m^*/6) & m^*(5/2 + m^*/12) & (1 + 3m^*/2 + m^{*2}/36) \end{pmatrix}$$

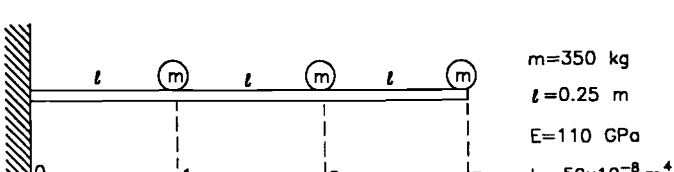

Fig. 9.11 Cantilever beam modeled as a lumped system.

Finally, the equation relating the state vector at 0 to that at 3 is

$$\begin{aligned} Z_3 &= A_3 Z_0 = A_1 A_1 A_1 Z_0 \\ A_3 &= A_1 A_1 A_1 \end{aligned} \tag{9.104}$$

The elements in A_3 are lengthy and not all are needed in the computations. The ones that are needed are worked out and given as follows:

$$a_{13} = 9/2 + m^* + m^{*2}/72$$

$$a_{14} = 9/2 + 16m^*/36 + m^{*2}/216$$

$$a_{33} = 1 + 3m^* + m^{*2}/12$$

$$a_{34} = 3 + 5m^*/3 + m^{*2}/36$$

$$a_{43} = 7m^* + 13m^{*2}/12 + m^{*3}/72$$

$$a_{44} = 1 + 6m^* + 17m^{*2}/36 + m^{*3}/216$$

The relationship between M_3^*, V_3^*, and M_0^*, V_0^* is

$$\begin{pmatrix} M_3^* \\ V_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} M_0^* \\ V_0^* \end{pmatrix} \tag{9.105}$$

Because M_0^* and V_0^* are not zero, the determinant of the coefficients must be zero, i.e.,

$$a_{33}a_{44} - a_{34}a_{43} = 0$$

When the preceding equation is expanded in detail, we find the expression for determining the natural frequencies as

$$26m^{*3} - 786m^{*2} + 2592m^* - 216 = 0$$

Three roots are obtained, and they are

$$m_1^* = 0.0855462, \qquad m_2^* = 3.66778, \qquad m_3^* = 26.4774$$

Newton's iteration method has been used in finding the preceding roots. The natural frequencies for principal modes are computed as follows. For m_1^*, we have

$$\omega_1^2 = \frac{EI}{m\ell^3} m_1^* = \frac{110 \times 10^9 \times 52 \times 10^{-8}}{350 \times (0.25)^3}(0.0855462) = 894.764$$

$$\omega_1 = 29.9126 \qquad (\text{s}^{-1})$$

With the value of m^* determined, we find the numerical values of a_{33} and a_{34}:

$$a_{33} = 1.257248, \qquad a_{34} = 3.427934$$

Then from Eq. (9.105), we have

$$M_0^* = -2.726538\ V_0^* \tag{9.106}$$

To determine the shape of the fundamental mode, we use Eqs. (9.102–9.104) and find

$$Y_1^* = \tfrac{1}{2}M_0^* + \tfrac{1}{6}V_0^* = -1.196602\ V_0^*$$

$$Y_2^* = (2 + m^*/12)M_0^* + (4/3 + m^*/36)V_0^* = -4.136803\ V_0^*$$

$$\begin{aligned} Y_3^* &= (4.5 + m^* + m^{*2}/72)M_0^* \\ &\quad + (4.5 + 16m^*/36 + m^{*2}/216)V_0^* = -7.964889\ V_0^* \end{aligned}$$

In common practice, the mode shape is expressed as a ratio of magnitudes. Let us compute the ratios and find

$$Y_1^*/Y_3^* = 0.150235$$

$$Y_2^*/Y_3^* = 0.519380$$

That means as $Y_3^* = 1$,

$$Y_1^* = 0.150235, \qquad Y_2^* = 0.519380 \tag{9.107}$$

For $m_2^* = 3.66778$, we get

$$\omega_2^2 = 38362.88$$

$$\omega_2 = 195.864 \qquad (\mathrm{s}^{-1})$$

$$Y_1^*/Y_3^* = -1.268889, \qquad Y_2^*/Y_3^* = -1.507484$$

Similarly, for $m_3^* = 26.4774$, we have

$$\omega_3^2 = 276938.47$$

$$\omega_3 = 526.249 \qquad (\mathrm{s}^{-1})$$

$$Y_1^*/Y_3^* = 4.647174, \qquad Y_2^*/Y_3^* = -3.248295$$

9.4 Vibrations of Continuous Systems

Many practical systems that we deal with every day are continuous in nature. Therefore, without studying the vibrations of continuous systems, the knowledge of vibration analysis will not be complete. In this section, we will study some simple cases such as vibrating string, beam, and membrane. The materials involved are assumed to be homogeneous, isotropic, and obeying Hooke's law in stress and strain relations. In addition, because sound waves are a vibration of continuous medium, they also will be studied in this section. From this section, the reader will learn fundamentals in setting up a partial differential equation and methods for solving them. A Fourier series will be used in the solutions of the problems presented in the examples.

Vibrating String

Before deriving a partial differential equation for a vibrating string, we make the following assumptions:

1) The string is perfectly flexible, that is, it cannot resist any bending moments.
2) The vertical deflection y of the string is small compared with the length L.
3) The slope at any point of the deflected string is small compared with unity.
4) The tension T is constant at all times and at all points of the deflected string, and is large compared with the weight of the string.
5) The horizontal displacement of the string is negligible compared to the vertical displacement, that is, we have pure transverse vibrations.
6) The motion takes place only in the x–y plane.

Consider that the string is fixed at the ends and subjected to a constant tension of T. Let us take a small segment $\mathrm{d}s$ of the string as shown in Fig. 9.12, and let w be the weight per unit length of the string. From $\sum F = ma$, and

$$\sum F_y = -T\sin\alpha + T\sin\beta - w\,\mathrm{d}s$$

we can set up the equation of motion. With the preceding assumptions, we have

$$\mathrm{d}y \ll \mathrm{d}x, \qquad \mathrm{d}s \simeq \mathrm{d}x$$

$$\sin\alpha \simeq \tan\alpha, \qquad \sin\beta \simeq \tan\beta$$

Because,

$$\tan\alpha = \frac{\partial y}{\partial x}, \qquad \tan\beta = \frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\mathrm{d}x$$

$$\sum F_y = T\frac{\partial^2 y}{\partial x^2}\mathrm{d}x - w\,\mathrm{d}x \tag{9.108}$$

On the other hand,

$$a_y = \frac{\partial^2 y}{\partial t^2}$$

$$ma_y = \frac{\omega}{g}\mathrm{d}x\frac{\partial^2 y}{\partial t^2} \tag{9.109}$$

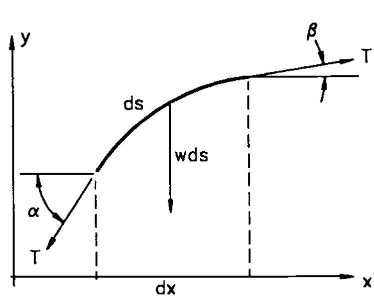

Fig. 9.12 Forces on a small segment of the string.

Combining Eqs. (9.108) and (9.109), we obtain

$$T\frac{\partial^2 y}{\partial x^2}\mathrm{d}x - w\,\mathrm{d}x = \frac{w}{g}\mathrm{d}x\frac{\partial^2 y}{\partial t^2}$$

or

$$\frac{gT}{w}\frac{\partial^2 y}{\partial x^2} - g = \frac{\partial^2 y}{\partial t^2}$$

Let $a^2 = gT/w$ and because of fourth assumption, we drop the term g on the left-hand side and find

$$a^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{9.110}$$

This is the partial differential equation for the transverse vibration of a string. It is also called the one-dimensional wave equation.

The boundary conditions for the case of vibrating string can be written as 1) $y(o,t) = 0$; 2) $y(L,t) = 0$; 3) $\partial y/\partial t(x,o) = g(x)$; and 4) $y(x,o) = f(x)$. Because the ends of the string are fixed, we have $y = 0$ at $x = 0$, and $x = L$ for all time t. The third and fourth conditions are the initial velocity and initial displacement of the string.

Here a reader may question the number of boundary conditions necessary for solving partial differential equations. In solving the ordinary differential equations, we know, in general, the number of boundary conditions equals the order of differential equations. Because one integral constant will appear when the equation is integrated once, such a constant must be determined by one boundary condition. In solving the partial differential equations, we may state that the number of boundary conditions needed for solving the problem equals the number of necessary conditions needed for determining the arbitrary functions after integrating the partial differential equation.

Solution of the Vibrating String with Initial Displacement

First let us consider the problem of the vibrating string with the initial displacement. Note that the equation of motion is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \tag{9.110}$$

where $a^2 = Tg/w$. The boundary conditions are

$$y(0,t) = 0, \qquad y(L,t) = 0$$

$$\frac{\partial y}{\partial t}(x,0) = 0, \qquad y(x,0) = f(x) \tag{9.111}$$

where $f(x)$ is known. To solve such a problem, we assume

$$y(x,t) = X(x)T(t) \tag{9.112}$$

This method is known as a separation of variables. Substituting this expression into Eq. (9.110) leads to

$$XT'' = a^2X''T$$

or

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T}$$

where $T'' = \mathrm{d}^2T/\mathrm{d}t^2$ and $X'' = \mathrm{d}^2X/\mathrm{d}x^2$.

Now the right-hand side of the preceding equation is independent of x and the left-hand side is independent of t. Because they are equal, their common value must be a constant, say λ. Hence

$$\frac{X''}{X} = \lambda, \qquad \frac{T''}{a^2T} = \lambda$$

or

$$X'' - \lambda X = 0, \qquad T'' - \lambda a^2 T = 0 \tag{9.113}$$

Thus, we have two ordinary differential equations. To satisfy the boundary conditions, the value of λ must be less than zero, i.e., $\lambda = -\beta^2$ where β is real. Hence Eqs. (9.113) become

$$X'' + \beta^2 X = 0, \qquad T'' + \beta^2 a^2 T = 0 \tag{9.114}$$

The solution then can be written as

$$y(x,t) = (A\cos\beta x + B\sin\beta x)(C\cos\beta at + D\sin\beta at)$$

where A, B, C, D, and β are to be determined. Applying the first boundary condition gives

$$0 = y(0,t) = AT(t)$$

Hence $A = 0$. Applying the second boundary condition leads to

$$0 = y(L,t) = (B\sin\beta L)T(t)$$

That means $\sin\beta L$ must be zero, so that

$$\beta L = \pm n\pi \quad n = 1, 2, 3, \ldots$$

or

$$\beta = \pm\frac{n\pi}{L}$$

The function $X(x)$ can be written in the form of

$$\begin{aligned} X(x) &= B_n^* \sin \frac{n\pi}{L}x + B_{-n}^* \sin\left(-\frac{n\pi}{L}\right) \\ &= (B_n^* - B_{-n}^*) \sin \frac{n\pi}{L}x \\ &= B_n \sin \frac{n\pi}{L}x \end{aligned}$$

so we just consider

$$\beta = \frac{n\pi}{L} \quad n = 1, 2, 3, \ldots$$

and

$$y_n(x, t) = B_n \sin \frac{n\pi}{L}x\left[c_n \cos \frac{n\pi}{L}at + D_n \sin \frac{n\pi}{L}at\right]$$

On the other hand, to determine the constants in $T(t)$, we apply the third boundary condition

$$\frac{\partial y}{\partial t}(x, 0) = 0$$

$$X(x)T'(0) = 0$$

$$T'(0) = -\beta aC \sin 0 + \beta aD \cos 0 = \beta aD = 0$$

Hence

$$D = 0$$

$$T(t) = C \cos \beta at$$

Because $\beta = n\pi/L$ and n is an integer, with $T_n(t)$ for a specific n, we have

$$T_n(t) = C_n \cos \frac{n\pi a}{L}t$$

Combining $X_n(x)$ and $T_n(t)$, we get

$$\begin{aligned} X_n(x)T_n(t) &= B_n \sin \frac{n\pi}{L}xC_n \cos \frac{n\pi a}{L}t \\ &= b_n \sin \frac{n\pi x}{L} \cos \frac{n\pi a}{L}t \end{aligned}$$

where $b_n = B_nC_n$. Now this is a solution of the partial differential equation and satisfies three boundary conditions for all n, where $n = 1, 2, 3, \ldots$. Because the wave equation is a linear partial differential equation which has the property that any linear combination of solutions is its solution. Thus, the general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}x \cos \frac{n\pi a}{L}t \tag{9.115}$$

To determine b_n, we apply the fourth boundary condition

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \tag{9.116}$$

Although $f(x)$ is defined in $0 \le x \le L$, because we are only interested in this interval, we can prolong the function in $(-L \le x \le 0)$ and consider it as a periodic odd function in the whole space, then with the use of Fourier sine series, b_n can be found as

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

Therefore the complete solution becomes

$$y(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left[\int_0^L f(z) \sin \frac{n\pi z}{L} dz \right] \sin \frac{n\pi x}{L} \cos \frac{n\pi a}{L} t \tag{9.117}$$

On the other hand with the use of the formula from trigonometry, we can write

$$\sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t = \frac{1}{2} \left[\sin \frac{n\pi}{L}(x - at) + \sin \frac{n\pi}{L}(x + at) \right]$$

then the solution becomes

$$y(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}(x - at) + \frac{1}{2} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}(x + at)$$

Because

$$f(z) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} z$$

we have

$$f(x - at) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}(x - at)$$

and

$$f(x + at) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}(x + at)$$

The solution is simply

$$y(x,t) = \tfrac{1}{2}[f(x - at) + f(x + at)] \tag{9.118}$$

Before looking into the physical meaning of the two functions $f(x - at)$ and $f(x + at)$, we first recall that they are the functions representing the vertical displacements of the string along the whole space of x from $-\infty$ to ∞ because they are derived from the Fourier series expansion of $f(x)$. Then we recognize that they are different from $f(x)$ given in Eq. (9.111), which is true only for x from 0 to L. The two functions $f(x - at)$ and $f(x + at)$ represent two waves traveling in opposite directions along the string, each with velocity a. To show this, we make the following observations.

Consider $f(x - at)$. At $t = 0$, $y(x) = 1/2f(x)$ is the half of the initial displacement. At any later time t_1, it defines the curve $1/2f(x - at_1)$. The two curves are identical except that the latter is translated to the right a distance at_1. Thus, the configuration moves along the string without distortion a distance at_1 in t_1 units of time. The velocity of this progression is therefore a.

Similarly the function $f(x + at)$ defines a configuration of $y(x) = 1/2f(x)$ that moves to the left along the string with constant velocity a. Hence, the entire configuration is the sum of the two functions.

Example 9.11

A string of length of 10 units is fixed at both ends and given the initial displacement as

$$f(x) = \frac{x(10 - x)}{1000} \qquad \text{for } 0 < x < 10 \tag{9.119}$$

It is released from rest. Assume that the string has $a^2 = 10,000$ units. Determine its subsequent motion.

Solution. According to the solution derived in the section, we assume that $f(x)$ is an odd periodic function as shown in Fig. 9.13, then we have

$$y(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \mathrm{d}x$$

$$y(x, t) = \frac{2}{L} \sum_{n=1}^{\infty} \left[\int_0^L f(z) \sin \frac{n\pi}{L} z \mathrm{d}z \right] \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t$$

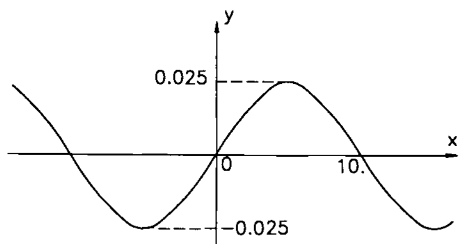

Fig. 9.13 Initial displacement of the string with an odd function assumed.

Evaluating b_n for the given $f(x)$, we find

$$b_n = \frac{2}{10}\int_0^{10} \frac{x(10-x)}{1000} \sin\frac{n\pi}{10}x\,dx$$

$$= \frac{2}{5(n\pi)^3}[1-(-1)^n] = \begin{cases} 0 & n = \text{even} \\ \frac{4}{5(n\pi)^3} & n = \text{odd} \end{cases}$$

Hence

$$y(x,t) = \frac{4}{5\pi^3}\sum_{n=1,3,5}^{\infty} \frac{1}{n^3}\sin\frac{n\pi}{10}x\cos(10n\pi t)$$

Note that in the process of deriving the solution of the wave equation, the initial displacement function has been assumed to be a periodic odd function. The given initial displacement, Eq. (9.118), is true only between two ends. If the expression $f(x)$ is true for all the values of x, then Eq. (9.117) can be used directly for the solution, which is illustrated in the following example.

Example 9.12

A string stretching to infinity in both directions is given the initial displacement

$$f(x) = \frac{1}{1+8x^2}$$

and released from rest. (One remark must be made here. In many engineering problems, the term "infinity" means that the boundary is far away from space reached by the motion. For this particular example, infinity means before a reflected wave is observed.) Find the displacement during its subsequent motion.

Solution. Using Eq. (9.117), we have the solution as simply

$$y(x,t) = \frac{1}{2}[f(x-at) + f(x+at)]$$

$$= \frac{1}{2}\left[\frac{1}{1+8(x-at)^2} + \frac{1}{1+8(x+at)^2}\right]$$

Solution of the Vibrating String with Initial Velocity and Displacement

Now let us consider the problem of the vibrating string stretching from $-\infty$ to ∞. Rewrite the equation of motion, Eq. (9.110), as

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

with the boundary conditions given as

$$y(x,0) = f(x), \qquad \frac{\partial y}{\partial t}(x,0) = g(x) \tag{9.120}$$

The general solution of Eq. (9.110) is

$$y(x,t) = y_1(x - at) + y_2(x + at)$$

where $y_1(x - at)$ and $y_2(x + at)$ are arbitrary. The task here is to relate these two functions with the given function $f(x)$ and $g(x)$. Applying Eqs. (9.120), we have

$$y(x,0) = f(x) = y_1(x) + y_2(x) \tag{A}$$

$$\frac{\partial y}{\partial t}(x,0) = g(x) = -ay_1'(x) + ay_2'(x) \tag{B}$$

Dividing the second of the preceding equations by a and then integrating, we find

$$-y_1(x) + y_2(x) = \frac{1}{a}\int_{x_0}^{x} g(x)\mathrm{d}x \tag{C}$$

Combining Eqs. (A) and (C), gives

$$y_1(x) = \frac{1}{2}\left[f(x) - \frac{1}{a}\int_{x_0}^{x} g(x)\mathrm{d}x\right] \tag{D}$$

$$y_2(x) = \frac{1}{2}\left[f(x) + \frac{1}{a}\int_{x_0}^{x} g(x)\mathrm{d}x\right] \tag{E}$$

With the form of y_1 and y_2 known, we can now write

$$\begin{aligned}
y(x,t) &= y_1(x - at) + y_2(x + at) \\
&= \frac{1}{2}\left[f(x - at) - \frac{1}{a}\int_{x_0}^{x-at} g(x)\mathrm{d}x\right] + \frac{1}{2}\left[f(x + at) + \frac{1}{a}\int_{x_0}^{X+at} g(x)\mathrm{d}x\right] \\
&= \frac{1}{2}\left[f(x - at) + f(x + at) + \frac{1}{a}\int_{x-at}^{x+at} g(x)\mathrm{d}x\right]
\end{aligned}$$

Transverse Vibrations of a Beam

Consider a beam of length L loaded by a variable load $w(x,t)$. To simplify the problem, assumptions are made as follows:

1) The weight of the beam is included in the load w.
2) The vertical deflection y is small compared to the length L.
3) The slope of the deflection curve is much smaller than unity.
4) The horizontal displacement of the beam is negligible compared to the vertical displacement; that is, we have pure transverse motion.
5) The assumptions for beam theory hold: Every layer of material is free to expand and contract longitudinally and laterally under stress as if it is separated from other layers; the tensile and compressive moduli of elasticity are equal; and the cross section remains a plane surface.

Now let us consider a small segment $\mathrm{d}s$ of a bent beam as shown in Fig. 9.14. Let e be the amount of length changed from its original length $\mathrm{d}s$ on the fiber UV.

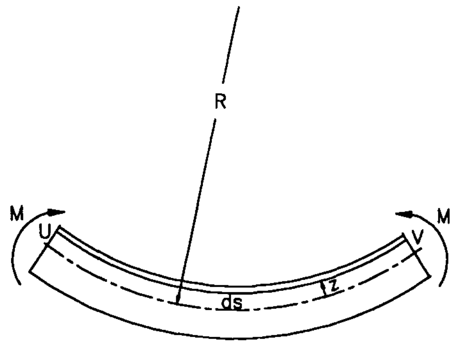

Fig. 9.14 Small segment of a beam.

We have

$$\frac{ds - e}{ds} = \frac{R - z}{R}$$

or

$$\frac{z}{R} = \frac{e}{ds}$$

where R is the radius of curvature of the deflection curve. The strain is defined positive for tension and negative for compression; thus,

$$\epsilon = -\frac{e}{ds} = -\frac{z}{R}$$

By using Hooke's law,

$$\sigma = E\epsilon = -\frac{zE}{R}$$

The force acting on the area dA is then

$$dF = \sigma \, dA = -\frac{Ez}{R} dA$$

Because the tensile and compressive forces are equal over any cross section, the total force acting over the whole cross section is zero:

$$F = -\int_A \frac{Ez}{R} \, dA = -\frac{E}{R} \int_A z \, dA = 0$$

This result means that the neutral axis passed through the centroid of the cross-sectional area. On the other hand, under equilibrium, the internal bending moment created by the stress σ must be the same as the external moment M:

$$M = \int_A (-z)\sigma \, dA = \frac{E}{R} \int_A z^2 dA = \frac{EI}{R}$$

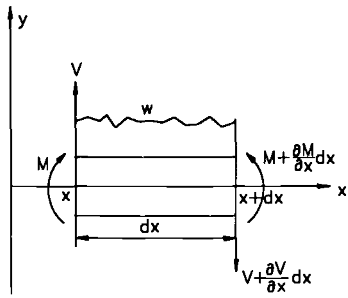

Fig. 9.15 Load on a segment of a beam.

where I is the moment of inertia of the area about the neutral axis. From studies in mathematics, we also learn that the curvature of a plane curve is given by the equation

$$\frac{1}{R} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} = \frac{d^2y}{dx^2}$$

because $dy/dx \ll 1$. Therefore

$$M = EI\frac{d^2y}{dx^2} \tag{A}$$

Referring to Fig. 9.15, we can compute the sum of the force in the y direction

$$\sum F_y = V - \left(V + \frac{\partial V}{\partial x}\,dx\right) - w\,dx = -\frac{\partial V}{\partial x}dx - w\,dx \tag{B}$$

and the inertial force is

$$ma_y = \frac{w}{g}dx\frac{\partial^2 y}{\partial t^2} \tag{C}$$

Equating Eq. (B) to (C), we find

$$-\frac{\partial V}{\partial x} - w = \frac{w}{g}\frac{\partial^2 y}{\partial t^2} \tag{D}$$

On the other hand, taking moments about the point x, we have

$$\sum M = M - \left(M + \frac{\partial M}{\partial x}dx\right) + \left(V + \frac{\partial V}{\partial x}dx\right)dx + w\,dx\frac{dx}{2}$$

$$= -\frac{\partial M}{\partial X}dx + V\,dx + \frac{\partial V}{\partial x}(dx)^2 + \frac{w}{2}(dx)^2$$

Neglecting high order terms and setting $\sum M = 0$ leads to

$$V = \frac{\partial M}{\partial x} \tag{E}$$

Substituting Eq. (E) into Eq. (D), we get

$$-\frac{\partial^2 M}{\partial x^2} - w = \frac{w}{g}\frac{\partial^2 y}{\partial t^2}$$

Using Eq. (A), then we obtain

$$-\frac{\partial^2}{\partial X^2}\left(EI\frac{\partial^2 y}{\partial X^2}\right) - w = \frac{w}{g}\frac{\partial^2 y}{\partial t^2} \tag{9.121}$$

This is the partial differential equation for the transverse vibration of a beam. Note that upward y is positive, but downward w is positive in Eq. (9.121). If we are interested only in studying the free vibration of the beam, the load term is dropped, and Eq. (9.121) becomes

$$\frac{w}{g}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial X^2}\right) = 0$$

The equation can be further simplified for $EI = \text{const}$:

$$\frac{\partial^2 y}{\partial t^2} + a^2\frac{\partial^4 y}{\partial x^4} = 0 \tag{9.122}$$

where $a^2 = EIg/w$.

Solving the partial differential equation, we must use some necessary boundary conditions. Boundary conditions for two popular beams follow.

1) Boundary conditions for simply supported beams:

$$y(0, t) = 0$$

$$y(L, t) = 0$$

$$\frac{\partial^2 y}{\partial x^2}(0, t) = 0 \qquad \text{for } M = 0 \qquad \text{at } x = 0$$

$$\frac{\partial^2 y}{\partial x^2}(L, t) = 0 \qquad \text{for } M = 0 \qquad \text{at } x = L$$

$$y(x, 0) = f(x)$$

$$\frac{\partial y}{\partial t}(x, 0) = g(x)$$

2) Boundary conditions for built-in beams:

$$y(0, t) = 0$$

$$y(L, 0) = 0$$

$$\frac{\partial y}{\partial x}(0, t) = 0 \qquad \text{for slope } = 0 \qquad \text{at } x = 0$$

$$\frac{\partial y}{\partial x}(L, t) = 0 \qquad \text{for slope } = 0 \qquad \text{at } x = L$$

$$y(x, 0) = f(x)$$

$$\frac{\partial y}{\partial t}(x, 0) = g(x)$$

Example 9.13

A simply supported beam is given the initial displacement $f(x)$ and released from rest. Determine its subsequent motion.

Solution. The conditions given establish Eq. (9.122) as the equation of motion; it is rewritten here for convenience:

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial x^4} = 0 \tag{9.122}$$

We shall seek a separable solution of the form

$$y(x, t) = X(x)T(t)$$

and we have

$$X \frac{d^2 T}{dt^2} + a^2 T \frac{d^4 X}{dx^4} = 0$$

or

$$-\frac{a^2}{X} \frac{d^4 X}{dx^4} = \frac{1}{T} \frac{d^2 T}{dt^2} \tag{9.123}$$

Because the left side of Eq. (9.123) is a function of x alone, and the right side is a function of t only, the common value for the equation must be a constant, say λ. Thus,

$$-\frac{a^2}{X} \frac{d^4 X}{dx^4} = \frac{1}{T} \frac{d^2 T}{dt^2} = \lambda$$

To satisfy the boundary conditions, it is found that λ must be negative. Let $\lambda = -\omega^2$, then we have two ordinary differential equations:

$$\frac{d^2 T}{dt^2} + \omega^2 T = 0 \tag{A}$$

$$\frac{d^4 X}{dx^4} - \frac{\omega^2}{a^2} X = 0 \tag{B}$$

The solution of Eq. (A) is known as

$$T(t) = C_1 \sin \omega t + C_2 \cos \omega t \tag{C}$$

The solution for Eq. (B) is assumed as

$$X(x) = Ae^{sx} \tag{D}$$

where A and s are constant. Substituting the assumed solution into Eq. (B) gives

$$\left(s^4 - \frac{\omega^2}{a^2}\right)Ae^{sx} = 0$$

From the equation we obtain the four roots

$$S_1 = \sqrt{\frac{\omega}{a}} = \alpha \qquad S_2 = -\sqrt{\frac{\omega}{a}} = -\alpha$$

$$S_3 = i\sqrt{\frac{\omega}{a}} = i\alpha \qquad s_4 = -i\sqrt{\frac{w}{a}} = -i\alpha$$

The solution is then

$$X(x) = A_1e^{\alpha x} + A_2e^{-\alpha x} + A_3e^{+i\alpha x} + A_4e^{-i\alpha x} \tag{E}$$

where A_1, A_2, A_3, and A_4 are arbitrary. Without loss of generality, we can write the solution as

$$X(x) = C_3 \sinh \alpha x + C_4 \cosh \alpha x + C_5 \sin \alpha x + C_6 \cos \alpha x$$

The solution of Eq. (9.122) is then

$$\begin{aligned} y(x, t) = (C_1 \sin \omega t + C_2 \cos \omega t)(C_3 \sinh \alpha x + C_4 \cosh \alpha x \\ + C_5 \sin \alpha x + C_6 \cos \alpha x) \end{aligned} \tag{F}$$

with

$$\omega = \alpha^2 a$$

The constants appearing in the solution and the natural frequencies are determined by applying the boundary conditions. For a simply supported beam, the boundary conditions are

$$y(0, t) = y(L, t) = 0$$

$$\frac{\partial^2 y}{\partial x^2}(0, t) = \frac{\partial^2 y}{\partial x^2}(L, t) = 0$$

Applying the boundary conditions to Eq. (F) gives

$$C_4 + C_6 = 0$$

$$C_3 \sinh \alpha L + C_4 \cosh \alpha L + C_5 \sin \alpha L + C_6 \cos \alpha L = 0$$

$$C_4 - C_6 = 0$$

$$C_3 \sinh \alpha L + C_4 \cosh \alpha L - C_5 \sin \alpha L - C_6 \cos \alpha L = 0$$

From the four preceding equations, we find $C_3 = C_4 = C_6 = 0$, and

$$C_5 \sin \alpha L = 0$$

Therefore, the natural frequencies can be determined from

$$\alpha L = n\pi \quad n = 1, 2, 3, \ldots$$

and we obtain

$$\omega_n = \alpha_n^2 a = \left(\frac{n\pi}{L}\right)^2 a = (n\pi)^2 \sqrt{\frac{EIg}{wL^4}} \tag{9.124}$$

With the natural frequencies determined, the general solution Eq. (F) becomes

$$y(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin \frac{n\pi}{L} x$$

where $A_n = (C_1 C_5)_n$ and $B_n = (C_2 C_5)_n$. The constants A_n and B_n can be determined by initial conditions of the motion. For this example, $y(x, 0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$$

By assuming $f(x)$ as a periodic odd function, we obtain

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, \mathrm{d}x$$

Because initial velocity is zero, A_n must be zero. Therefore, the complete solution is

$$y(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left[\int_0^L f(z) \sin \frac{n\pi z}{L} \mathrm{d}z \right] \sin \frac{n\pi x}{L} \cos \omega_n t \tag{9.125}$$

Example 9.14

A simply supported beam of length L is subjected to a concentrated harmonic force $F_0 \sin \omega_f t$ as shown in Fig. 9.16. Determine its subsequent motion.

Solution. The governing equation is

$$EI \frac{\partial^4 y}{\partial x^4} + \frac{w}{g} \frac{\partial^2 y}{\partial t^2} = F_0 \sin \omega_f t \delta(x - a) \tag{A}$$

To find the response of the forced vibration, we consider the forcing function as

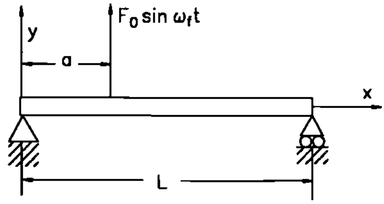

Fig. 9.16 Concentrated harmonic force acting on a simple beam.

an odd periodic function with period of $2L$ as shown in Fig. 9.17, and expand the function into a Fourier sine series as

$$\delta(x-a) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

where

$$b_n = \frac{2}{L} \int_0^L \delta(x-a) \sin \frac{n\pi}{L} x \, dx = \frac{2}{L} \sin \frac{n\pi a}{L}$$

Therefore Eq. (A) becomes

$$EI \frac{\partial^4 y}{\partial x^4} + \frac{w}{g} \frac{\partial^2 y}{\partial t^2} = \frac{2F_0}{L} \left(\sum_{n=1}^{\infty} \sin \frac{n\pi a}{L} \sin \frac{n\pi}{L} x \right) \sin \omega_f t \tag{B}$$

To find the forced response, we assume the solution as

$$y(x,t) = f(x) \sin \omega_f t \tag{C}$$

Substituting Eq. (C) into Eq. (B) gives

$$EI \frac{d^4 f}{dx^4} - \frac{w}{g} \omega_f^2 f = \frac{2F_0}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi a}{L} \sin \frac{n\pi}{L} x \tag{D}$$

where the common factor $\sin \omega_f t$ on both sides of the equation has been dropped. The particular solution of Eq. (D) is assumed as

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

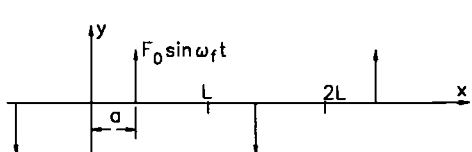

Fig. 9.17 Concentrated force assumed as a periodic odd function of *x*.

With this, Eq. (D) becomes

$$\frac{\pi^4 EI}{L^4}\sum_{n=1}^{\infty} n^4 A_n \sin\frac{n\pi x}{L} - \frac{w}{g}\omega_f^2 \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

$$= \frac{2F_0}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi a}{L}\sin\frac{n\pi x}{L}$$

Equating coefficients of $\sin(n\pi x/L)$ gives

$$A_n = \frac{2F_0 L^3}{n^4\pi^4 EI - \frac{w}{g}L^4\omega_f^2}\sin\frac{n\pi a}{L} \tag{9.126}$$

Therefore the forced response is obtained as

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\sin\omega_f t \tag{9.127}$$

From the denominator of Eq. (9.126), we find that a resonant condition is

$$\omega_f = n^2\pi^2\sqrt{\frac{EIg}{wL^4}}$$

A few remarks must be made before we end the section. In this analysis, the mass of the beam is considered in the inertial force, but the weight of the beam is neglected in the load. This means that the initial deflection caused by weight is small compared to the dynamic deflection. Examples given are solved successfully. The beam involved is supported simply. For other end conditions the solutions may become complicated. To fully understand the subject, additional references will be needed.

Vibration of a Circular Membrane

Suppose that a piece of membrane is mounted on a drum. The tension in the membrane is shown in Fig. 9.18. Our first task is to find the equation of motion for the vibrating membrane. To simplify the considerations, assumptions are made as follows.

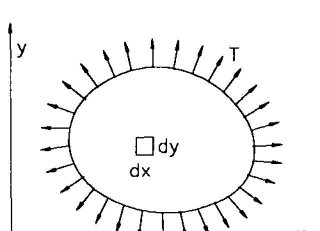

Fig. 9.18 Membrane tension.

1) Tension measured as force per unit length is normal to the boundary of the element and is constant throughout the membrane.

2) The total tension on the boundary is large compared to the weight of the membrane.

3) The membrane is so thin that it cannot resist any bending moment, i.e., there is no bending stress.

4) The vertical deflection w is small compared to the diameter of the membrane.

5) The slopes of the deflection surface are small compared to unity.

6) The lateral displacements are negligible compared with the vertical displacements.

Consider a differential element of the membrane with area $dxdy$. To analyze the force acting on this element, let us enlarge the element as shown in Fig. 9.19. Here P is applied pressure. The sum of forces in the z direction then can be computed as

$$\sum F_z = -T\,dy \tan\alpha + T\,dy \tan\beta - T\,dx \tan\gamma + T\,dx \tan\delta + P\,dx\,dy \quad \text{(A)}$$

Because slopes are small, the following relations have been used in the preceding equation:

$$\sin\alpha \simeq \tan\alpha, \qquad \sin\beta \simeq \tan\beta$$

$$\sin\gamma \simeq \tan\gamma, \qquad \sin\delta \simeq \tan\delta$$

Because w is the vertical displacement of the membrane, in the xz plane we have

$$\tan\alpha = \frac{\partial w}{\partial x}$$

$$\tan\beta = \frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) dx$$

Similarly in the yz plane, we have

$$\tan\gamma = \frac{\partial w}{\partial y}$$

$$\tan\delta = \frac{\partial w}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) dy$$

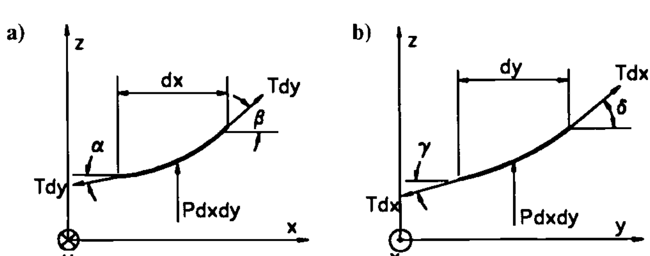

Fig. 9.19 Forces on a segment of membrane.

Substituting these expressions into Eq. (A) gives

$$\sum F_z = T\,\mathrm{d}x\,\mathrm{d}y\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + P\,\mathrm{d}x\,\mathrm{d}y \tag{B}$$

On the other hand, the mass of the element is

$$m = \rho\,\mathrm{d}x\,\mathrm{d}y \tag{C}$$

and the acceleration is

$$a_z = \frac{\partial^2 w}{\partial t^2} \tag{D}$$

where ρ is the mass per unit area of the membrane. The equation of motion then can be written as

$$\rho\,\mathrm{d}x\,\mathrm{d}y\frac{\partial^2 w}{\partial t^2} = T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\mathrm{d}x\,\mathrm{d}y + P\,\mathrm{d}x\,\mathrm{d}y$$

or

$$\frac{\partial^2 w}{\partial t^2} = a^2\nabla^2 w + \frac{1}{\rho}P(x, y) \tag{9.128}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad a^2 = \frac{T}{\rho}$$

Equation (9.128) can be applied to cylindrical coordinates that require the expression of ∇^2 as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

For the study of free vibration, the pressure term is dropped, and Eq. (9.128) becomes

$$\frac{\partial^2 w}{\partial t^2} = a^2\nabla^2 w \tag{9.129}$$

For simplicity, we consider a special case, that is, the membrane is initially deflected into a radially symmetrical form and is released from rest. The equation of motion is reduced to

$$\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{2}\frac{\partial w}{\partial r}\right) \tag{9.130}$$

And the boundary conditions are

$$
\begin{aligned}
w(R, t) &= 0 \\
w(r, 0) &= f(r) \\
\frac{\partial w}{\partial t}(r, 0) &= 0
\end{aligned}
\tag{9.131}
$$

where $r = R$ is the boundary of the membrane. Assume the solution as

$$w(r, t) = R(r)T(t) \tag{A}$$

Substituting the expression into Eq. (9.130) gives

$$a^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right) = \frac{T''}{T} = -\omega^2$$

where $R'' = (\mathrm{d}^2R/\mathrm{d}r^2)$, $R' = (\mathrm{d}R/\mathrm{d}r)$, $T'' = (\mathrm{d}^2T/\mathrm{d}t^2)$, and ω^2 is the arbitrary real constant. From which we get two ordinary differential equations as

$$rR'' + R' + \left(\frac{\omega}{a}\right)^2 rR = 0 \tag{B}$$

and

$$T'' + \omega^2 T = 0 \tag{C}$$

Equation (B) is known as Bessel's equation of order 0 with a parameter ω/a. The solution of Bessel's equation is

$$R(r) = AJ_0\left(\frac{\omega r}{a}\right) + BY_0\left(\frac{\omega r}{a}\right) \tag{D}$$

Because Y_0 approaches infinity as $r \to 0$, B must be zero. The solution of Eq. (C) is

$$T(t) = C\cos\omega t + D\sin\omega t \tag{E}$$

Combining Eqs. (D) and (E), we find

$$w(r, t) = J_0\left(\frac{\omega r}{a}\right)(C\cos\omega t + D\sin\omega t)$$

Because the initial velocity is zero, we have

$$D = 0.$$

Applying the first boundary condition gives

$$w(R, 0) = 0 = J_0\left(\frac{\omega R}{a}\right)$$

From this, the natural frequencies are determined. For example, the smallest root of $J_0 = 0$ is

$$\frac{\omega_1 R}{a} = 2.405$$

Hence, in general, we can write

$$J_0\left(\frac{\omega_n R}{a}\right) = 0 \quad n = 1, 2, 3, \ldots$$

for all the natural frequencies. The general solution becomes

$$w(r, t) = \sum_{n=1}^{\infty} C_n J_0\left(\frac{\omega_n r}{a}\right) \cos \omega_n t \tag{9.132}$$

To determine C_n, we apply the boundary condition Eq. (9.131)

$$w(r, 0) = f(r) = \sum_{n=1}^{\infty} C_n J_0\left(\frac{\omega_n r}{a}\right)$$

And with the use of the properties given in Appendix H, we find

$$C_n = \frac{2}{R^2[J_1(\omega_n R/a)]^2} \int_0^R r f(r) J_0\left(\frac{\omega_n r}{a}\right) \mathrm{d}r \tag{9.133}$$

Example 9.15

A circular membrane is fixed on its edge and given an initial displacement as

$$f(r) = 1 - r^2/100$$

It is released from rest. Assume that the diameter of the membrane is 20 units and the property of the membrane has $a^2 = 10{,}000$ units. Determine its subsequent motion.

Solution. From Eq. (9.133) we can compute the coefficients as

$$C_n = \frac{2}{R_1^2[J_1(\omega_n R/a)]^2} \int_0^R r \left[1 - \frac{r^2}{100}\right] J_0\left(\frac{\omega_n r}{a}\right) \mathrm{d}r$$

From the properties of Bessel functions given in Appendix F, we have

$$\int_0^L x J_0(x) \mathrm{d}x = L J_1(L)$$

$$\int_0^L x^3 J_0(x) \mathrm{d}x = L^3 J_1(L) - 2L^2 J_2(L)$$

Hence

$$C_n = \frac{2}{R^2[J_1(\omega_n R/a)]^2}\left\{\left(\frac{aR}{\omega_n}\right)J_1\left(\frac{\omega_n R}{a}\right)\right.$$

$$\left.-\frac{1}{100}\left[\left(\frac{aR^3}{\omega_n}\right)J_1\left(\frac{\omega_n R}{a}\right)-2\left(\frac{aR}{\omega_n}\right)^2 J_2\left(\frac{\omega_n R}{a}\right)\right]\right\}$$

$$= \frac{2}{[J_1(\omega_n R/a)]^2}\left\{\left[\left(\frac{a}{\omega_n R}\right)-\left(\frac{a}{\omega_n R}\right)\frac{R^2}{100}\right]\right.$$

$$\left.\times J_1\left(\frac{\omega_n R}{a}\right)+\frac{1}{50}\left(\frac{a}{\omega_n}\right)^2 J_2\left(\frac{\omega_n R}{a}\right)\right\}$$

$$= \frac{J_2(\omega_n R/a)}{25[J_1(\omega_n R/a)]^2}\left(\frac{a}{\omega_n}\right)^2$$

We determine ω_n from $J_0(\omega_n R/a) = 0$. Then with the use of the table of Bessel functions (Appendix H), we find

$$C_1 = 1.81152 \qquad \omega_1 = 24.05\ (\mathrm{s}^{-1})$$

$$C_2 = -0.139890 \qquad \omega_2 = 55.20\ (\mathrm{s}^{-1})$$

$$C_3 = 0.0455503 \qquad \omega_3 = 86.54\ (\mathrm{s}^{-1})$$

The solution then can be written as

$$w(r,t) = C_1 J_0\left(\frac{\omega_1 r}{a}\right)\cos\omega_1 t + C_2 J_0\left(\frac{\omega_2 r}{a}\right)\cos\omega_2 t$$

$$+C_3 J_0\left(\frac{\omega_3 r}{a}\right)\cos\omega_3 t + \cdots \tag{9.134}$$

Sound Waves in Fluid

Sound waves in air or water are longitudinal pressure waves propagating under an isentropic process. As the sound wave propagates, the change in pressure is small compared with the ambient pressure. Because of isentropic process, the change in density of the fluid is small compared with the original density. Because the viscous force plays no role in the sound wave, the equations involved in the phenomena are the continuity and the momentum equations only. These can be written as follows:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{V}) = 0 \tag{9.135}$$

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\nabla\boldsymbol{V} = -\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_s\nabla\rho = -a^2\left(\frac{\nabla p}{\rho}\right) \tag{9.136}$$

where $a = \sqrt{(\mathrm{d}P/\mathrm{d}\rho)_s}$ is the propagating speed of the sound wave. Based on the facts observed, we can express

$$\rho = \rho_0 + \epsilon\rho_1, \qquad V = \epsilon V_1 \tag{A}$$

where $\epsilon \ll 1$. From Eq. (9.135) we have to the ϵ order

$$\frac{\partial\rho_1}{\partial t} + \rho_0\nabla\cdot V_1 = 0 \tag{B}$$

From Eq. (9.136) we obtain, also to the ϵ order,

$$\frac{\partial V_1}{\partial t} = -a^2\frac{\nabla\rho_1}{\rho_0} \tag{C}$$

Differentiating Eq. (B) with respect to time t and substituting Eq. (C) into it gives

$$\frac{\partial^2\rho_1}{\partial t^2} - a^2\nabla^2\rho_1 = 0 \tag{9.137}$$

This is known as a wave equation. We have studied it in rectangular and cylindrical coordinates. Now let us study the wave equation in spherical coordinates such that

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{9.138}$$

However, to simplify the mathematics, we study a special case that is spherically symmetric, so that Eq. (9.137) becomes

$$\frac{\partial^2\rho_1}{\partial t^2} - a^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\rho_1}{\partial r}\right) = 0 \tag{9.139}$$

This equation can be rearranged to

$$\frac{\partial^2 r\rho_1}{\partial t^2} = a^2\frac{\partial^2 r\rho_1}{\partial r^2} \tag{9.140}$$

The solution of the preceding equation, similar to Eq. (9.117), can be written as

$$r\rho = f(r - at) + F(r + at) \tag{9.141}$$

As in the case of one-dimensional rectangular coordinates, the first term represents a wave advancing in the direction of r increasing, that is to say, a divergent wave, and the second term represents a converging wave. The latter does not possess much interest. To illustrate the physical meaning of the solution, let us consider the following example.

Example 9.16

Suppose that fireworks explode in the air; the initial change in density is

$$\Delta\rho = b \qquad \text{as } r < r_0$$

$$\Delta\rho = 0 \qquad \text{as } r > r_0$$

Determine the density change in the air during the propagating of the wave.

Solution. This is a case of divergent wave. Hence, only the first term in Eq. (9.141) is to be considered. The change in density of air is simply

$$\Delta\rho = b/r \qquad \text{as } 0 < r - at < r_0 \tag{9.142}$$

$$\Delta\rho = 0 \qquad \text{as } r - at < 0 \qquad \text{and} \qquad r - at > r_0 \tag{9.143}$$

This means that the higher density occurs in the spherical shell with the origin of the sphere where the fireworks exploded and with the thickness of r_0. This change in density is inversely proportional to the radius of the sphere. In other words, it will vanish as r approaches the infinite. The sphere is bounded by the radius of $(r_0 + at)$. Outside the sphere, there is no change in density. Also, the change vanishes as $r < at$. That is why the sound of the explosion can be heard only for a brief moment.

9.5 Nonlinear Vibrations

So far, we have studied many vibrating systems with linear characteristics. In discussing these systems, it was assumed that the force in a spring is proportional to the deformation. It was assumed also that, in the case of damping, the frictional force is a linear function of the velocity of motion. As a result of these assumptions, we had vibration systems represented by linear differential equations. However, there are practical problems in which these assumptions are no longer satisfactory to describe the actual motions. Such systems are called systems with nonlinear characteristics and are represented by nonlinear differential equations. In this section, we will deal with nonlinear vibration systems.

We may recall that the difference between a linear and a nonlinear differential equation is quite simple. If a differential equation contains products of unknown variables or products of unknown variable with the derivatives of unknown variables, the equation is nonlinear. Otherwise, the equation is linear.

As we learned in Chapter 8 and previous sections of this chapter, there are many analytical methods for solving linear differential equations. Because the principle of superposition is applicable to a linear equation, its general solution is the combination of all possible solutions.

For nonlinear differential equations, however, there are no definite methods for solving them analytically. Small perturbation methods may be considered as the systematic approach for solving them. One of the small perturbation methods, which is commonly used, has been introduced in Chapter 5 and will not be repeated here. On the other hand, because of the advancement of computer technology, many nonlinear problems whose solutions are not possible many years ago can be solved now. The following example illustrates this point.

Example 9.17

While a shaft is rotating at a high speed, the centrifugal force produced by the unbalanced disk can pull the shaft to a bow shape. This motion is known as the whirling of a rotating shaft. The sketch of the system is shown in Fig. 9.20. To consider the major dynamic properties of the motion, make some necessary assumptions and determine the equations of motion for the shaft rotating with and without acceleration; also find the maximum deflections for the two different conditions.

Solution. The following assumptions are made for this analysis.

1) The disk is rigid and is always perpendicular to the shaft.

2) The mass of the shaft is neglected.

3) Inertial forces lie in the plane of symmetry perpendicular to the shaft.

4) Damping is present and is assumed to be directly proportional to the precession speed of the shaft.

5) The supports are rigid, and the bearing flexibilities are neglected.

6) A torsional deformation is present, but vibration due to torsion will not be considered.

The geometry of the system is described as follows: The center of the mass of the disk is at point G at a distance e away from the center s of the shaft. The point o is the intersection of the straight line connecting two supports and the plane of symmetry. The center s is away from point o by a distance of r. The angle θ between line os and the reference line is the precession angle of the shaft, and $\dot{\theta}$

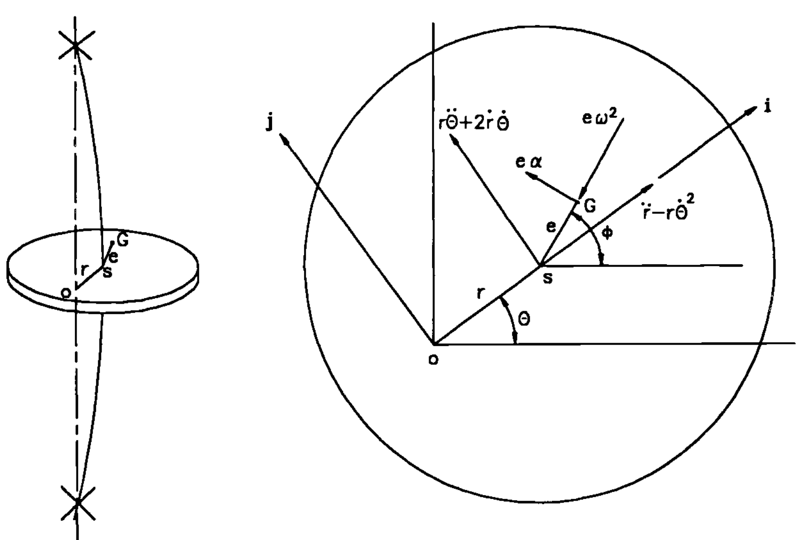

Fig. 9.20 Geometry of a whirling shaft.

is the precession speed that is considered to be different from the rotating speed ω of the shaft. The formulations for the two cases are considered separately.

Shaft rotating at a constant speed. It is known that the acceleration of point G relative to a fixed coordinate system can be expressed as

$$\boldsymbol{a}_G = \boldsymbol{a}_s + \boldsymbol{a}_{G/s} \tag{9.144}$$

where $\boldsymbol{a}_s$ is the acceleration of point s relative to point o and $\boldsymbol{a}_{G/s}$ is the relative acceleration between point G and point s. As the acceleration components are expressed along radial and tangential directions, they are found to be

$$\begin{aligned} \boldsymbol{a}_G = {} & [(\ddot{r} - r\dot{\theta}^2) - e\omega^2 \cos(\omega t - \theta)]\boldsymbol{i} \\ & + [(r\ddot{\theta} + 2\dot{r}\dot{\theta}) - e\omega^2 \sin(\omega t - \theta)]\boldsymbol{j} \end{aligned} \tag{9.145}$$

The equations of motion then can be written in the radial and tangential directions as

$$-kr - c\dot{r} = m[\ddot{r} - r\dot{\theta} - e\omega^2 \cos(\omega t - \theta)]$$

$$-cr\dot{\theta} = m[r\ddot{\theta} + 2\dot{r}\dot{\theta} - e\omega^2 \sin(\omega t - \theta)]$$

which can be rewritten into a familiar form of

$$\ddot{r} + c\dot{r}/m + (k/m - \dot{\theta}^2)r = e\omega^2 \cos(\omega t - \theta) \tag{9.146}$$

$$r\ddot{\theta} + (cr/m + 2\dot{r})\dot{\theta} = e\omega^2 \sin(\omega t - \theta) \tag{9.147}$$

Shaft rotating with acceleration. While the shaft is rotating with an angular acceleration α, additional acceleration in the tangential direction must be taken into account in considering $\boldsymbol{a}_{G/s}$. The acceleration of point G relative to a fixed system becomes

$$\begin{aligned} \boldsymbol{a}_G = {} & [\ddot{r} - r\dot{\theta}^2 - e\omega^2 \cos(\phi - \theta) - e\alpha \sin(\phi - \theta)]\boldsymbol{i} \\ & + [r\ddot{\theta} + 2\dot{r}\dot{\theta} - e\omega^2 \sin(\phi - \theta) + e\alpha \cos(\phi - \theta)]\boldsymbol{j} \end{aligned}$$

where ϕ is the rotating angular displacement of the shaft. The equations of motion for describing the whirling of the shaft become

$$-kr - c\dot{r} = m[\ddot{r} - r\dot{\theta}^2 - e\omega^2 \cos(\phi - \theta) - e\alpha \sin(\phi - \theta)] \tag{9.148}$$

$$-cr\dot{\theta} = m[r\ddot{\theta} + 2\dot{r}\dot{\theta} - e\omega^2 \sin(\phi - \theta) + e\alpha \cos(\phi - \theta)] \tag{9.149}$$

Although the effect of the angular acceleration α to the motion of whirling is to be explored, it is reasonable to simplify the considerations by setting $\omega = \alpha t$ and $\phi = \alpha t^2/2$. By doing these, Eqs. (9.148) and (9.149) become

$$-kr - c\dot{r} = m[\ddot{r} - r\dot{\theta}^2 - e(\alpha t)^2 \cos(\alpha t^2/2 - \theta) - e\alpha \sin(\alpha t^2/2 - \theta)] \tag{9.150}$$

$$-cr\dot{\theta} = m[r\ddot{\theta} + 2\dot{r}\dot{\theta} - e(\alpha t)^2 \sin(\alpha t^2/2 - \theta) + e\alpha \cos(\alpha t^2/2 - \theta)] \tag{9.151}$$

Maximum deflection as shaft rotates at a constant speed. Equations (9.146) and (9.147) are nonlinear equations of $r(t)$ and $\theta(t)$. The exact solution can only be obtained numerically. Before solving them it is proper to convert the variables into dimensionless forms. Let

$$r^* = r/e \tag{9.152a}$$

$$t^* = \omega_n t \tag{9.152b}$$

and

$$c = c_c\zeta = 2m\omega_n\zeta \tag{9.152c}$$

where c_c is the critical damping coefficient, $\omega_n = \sqrt{k/m}$, and ζ is the damping ratio. By introducing the preceding dimensionless variables, Eqs. (9.146) and (9.147) become

$$\ddot{r}^* + 2\zeta\dot{r}^* + (1 - \dot{\theta}^{*^2})r^* = (\omega/\omega_n)^2\cos(\omega t^*/\omega_n - \theta) \tag{9.153a}$$

$$\ddot{\theta}^* + 2(\zeta + \dot{r}^*/r^*)\dot{\theta}^* = (\omega/\omega_n)^2/r^*\sin(\omega t^*/\omega_n - \theta) \tag{9.153b}$$

These equations can be solved numerically with the use of the Runge–Kutta method. However, for a special case, as the whirling speed $\dot{\theta}$ is equal to the rotating speed ω of the shaft, it is called the synchronous whirl. Thus we have

$$\dot{\theta}^* = \omega/\omega_n \tag{9.154}$$

Under this condition,

$$\ddot{\theta}^* = \ddot{r}^* = \dot{r}^* = 0, \qquad \theta = (\omega/\omega_n)t^* + \beta \tag{9.155}$$

Equations (9.151a) and (9.151b) reduce to

$$[1 - (\omega/\omega_n)^2]r^* = (\omega/\omega_n)^2\cos\beta \tag{9.156}$$

$$2\zeta(\omega/\omega_n)r^* = (\omega/\omega_n)^2\sin\beta \tag{9.157}$$

where β is the phase angle between θ and ωt. Squaring Eqs. (9.156) and (9.157) and adding them together, we find

$$r^* = \frac{(\omega/\omega_n)^2}{\{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2\}^{1/2}} \tag{9.158}$$

Here we easily can see the maximum deflection increases as ω approaches ω_n. The numerical solution has been obtained by Ying.* It is lengthy. Details are revealed in the reference.

*Ying, S. J., "Transient Whirling of a Rotating Shaft with an Unbalanced Disk," *Rotating Machinery Dynamics*, ASME Pub. H0400B, Vol. 2, pp. 537–543, Sept. 1987.

Maximum deflection as shaft rotates with a constant acceleration. The equations of motion for describing the whirling of a shaft rotating with acceleration are given in Eqs. (9.150) and (9.151). By introducing the dimensionless quantities given in Eqs. (9.152a–9.152c), Eqs. (9.150) and (9.151) become

$$-r^* - 2\zeta\dot{r}^* = \ddot{r}^* - (\alpha^* t^*)^2 \cos(\alpha^* t^{*2}/2 - \theta) - \alpha^* \sin(\alpha^* t^{*2}/2 - \theta) \quad (9.159)$$

$$-2\zeta r^*\dot{\theta}^* = r^*\ddot{\theta}^* + 2\dot{r}^*\dot{\theta}^* - (\alpha^* t^*)^2 \sin(\alpha^* t^{*2}/2 - \theta) + \alpha^* \cos(\alpha^* t^{*2}/2 - \theta) \quad (9.160)$$

where $\alpha^* = \alpha/\omega_n^2$. Equations (9.159) and (9.160) are solved numerically by the Runge–Kutta method as given in Appendix A. The initial conditions are chosen as follows:

$$r^*(0) = 0.001, \qquad \theta(0) = 0, \qquad \dot{r}^*(0) = 0, \qquad \dot{\theta}^*(0) = 1.0$$

Because it is interesting to see the growth of r^* as the shaft rotates, the value of $r^*(0)$ should be as small as possible. However, a low $r^*(0)$ value could cause instability in the numerical computations. The term $r^*(0) = 0.001$ is a compromised quantity. The increment of time Δt^* used in the computation is 0.001, which satisfies the convergence criterion in all the cases calculated because further decrease in Δt^* does not change the results significantly. On the other hand, the range of time in the calculation is determined as follows. It is reasonable to assume that the maximum deflection will reach the peak in the range $0 < \omega/\omega_n < 3$. In all of the calculations, the number of maximum time steps is limited by $\omega/\omega_n = 3$. That is

$$\alpha t_{\max}/\omega_n = 3$$

or

$$t^*_{\max} = 3/\alpha^*$$

In this way $t^*_{\max}$ is determined for each value of α^* assigned. For example, as $\alpha^* = 0.01$, 300, 000 steps are calculated for the determination of maximum dimensionless deflection $R^*_{\max}$, and for $\alpha^* = 0.50$, 6000 steps are calculated. To find the effect of acceleration on the motion of rotating shaft, the range of α^* used for calculations is from 0.01 to 0.59 with an increment of 0.01. The results of the maximum dimensionless deflection vs dimensionless acceleration are plotted by a computer and are given in Fig. 9.21. From the curves shown in the figure, it easily is seen that for low damping factors $\zeta < 0.2$ the values of maximum deflection are higher at low acceleration. That means that while the shaft is rotating with low acceleration, the system has more time to stay in the neighborhood of resonance and $R^*_{\max}$ is occurring at low value of α^*. For systems with high damping factors $\zeta > 0.2$, the magnitude of whirling increases slightly with α^*. This is caused by the fact that the inertial force is not enough to overcome the damping force at low values of α^*.

Therefore, for slightly damped cases ($\zeta < 0.2$) the shaft should be operated with its highest possible acceleration to reach its operational speed; on the contrary, for

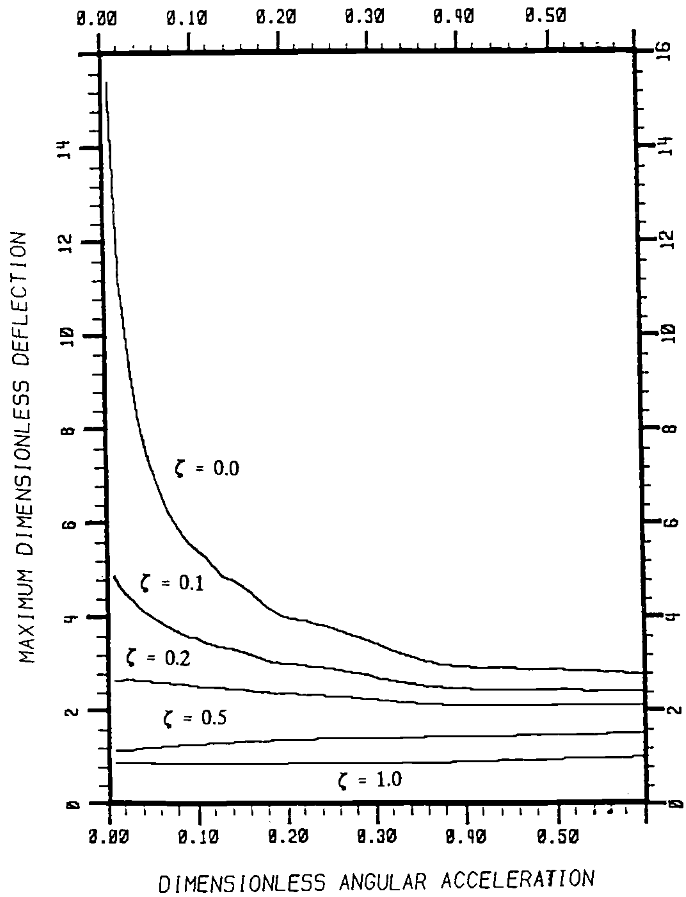

Fig. 9.21 Maximum deflection vs angular acceleration.

highly damped systems ($\zeta > 0.2$) the shaft should be operated with the lowest possible acceleration to reach its operational speed.

9.6 Stability of Vibrating Systems

Stability analysis is important in the study of vibrating systems. From the result of analysis, we can predict whether the amplitude of vibration will grow with time or not. For linear systems, we can determine the stability from the roots of characteristic equations. If the real parts of the roots are negative, the amplitudes of oscillations will decrease exponentially with the time; the system is stable. If the real parts are zero, then the harmonic motion will continue indefinitely, and the motion is still stable. However, for nonlinear systems, there is no characteristic equation so that we cannot predict the stability from the roots of characteristic equation. We must take a different approach for the analysis. Furthermore, we know that there is no analytical method to find the exact response of a nonlinear system. The following is the introduction of this new stability analysis. First we need to learn some new terminologies, and then we can discuss new concepts.

Phase Plane

The differential equation describing a nonlinear system may have the general form of

$$\ddot{x} + f(\dot{x}, x, t) = 0 \tag{9.161}$$

where the function f contains at least one term of the product of $x, \dot{x}$, or x and $\dot{x}$, such as $x^2, \dot{x}^2$, or $x\dot{x}$. If the function does not have the time t explicitly stated in the expression, then the system is known as an autonomous system that will be discussed in this section, and Eq. (9.159) becomes

$$\ddot{x} + f(\dot{x}, x) = 0 \tag{9.162}$$

In the study of stability, we define

$$\dot{x} = y \tag{9.163a}$$

$$\dot{y} = -f(x, y) \tag{9.163b}$$

Equation (9.163b) is actually the new form of Eq. (9.160). Consider x and y as the Cartesian coordinates. The x–y plane is called the phase plane.

Dividing Eq. (9.161b) by Eq. (9.161a), we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{f(x, y)}{y} \tag{9.164}$$

Integrating the equation gives

$$y = g(x)$$

which can be plotted in the phase plane and is called the trajectory. If the trajectory is bounded by a circle with finite radius, then x and y are limited; the system is stable. If at some points, $y = 0$ and $f(x, y) = 0$, the slope is indeterminate. We define such a point as a singular point. Further discussion will be presented for the integration of Eq. (9.164) around the singular point to determine whether the system is stable or unstable.

Example 9.18

Consider a simple pendulum. The differential equation of motion can be written as

$$\ddot{\theta} + \omega^2 \sin\theta = 0 \tag{9.165}$$

where $\omega^2 = g/L$, g is gravitational acceleration, and L is the length of the pendulum. Find the function for the trajectory. In the process of integration, an arbitrary constant will be present. Plot the trajectories in the phase plane for different arbitrary constants. Discuss whether the system is stable or unstable.

Solution. Let

$$\theta = x, \qquad \dot{\theta} = y$$

$$\omega^2 \sin\theta = \omega^2 \sin x = f(x, y)$$

then

$$\dot{y} = -\omega^2 \sin x$$

$$\frac{dy}{dx} = -\frac{f(x, y)}{y} = -\frac{\omega^2 \sin x}{y}$$

$$y\,dy = -\omega^2 \sin x\,dx$$

$$\tfrac{1}{2}y^2 + \omega^2(1 - \cos x) = E \tag{9.166}$$

where E is the arbitrary constant to be determined by the initial conditions. Note that E is proportional to the total energy of the system.

Equation (9.166) is the equation for trajectories. Three different trajectories are plotted as shown in Fig. 9.22 for $E = \omega^2, 2\omega^2, 3\omega^2$ with $\omega^2 = 1$. We notice that for $E < 2\omega^2$ we obtain closed trajectories, so that the motion repeats itself. This implies that the motion is stable. For $E > 2\omega^2$, the trajectories are open and the motion is unstable with the pendulum going over the top. The trajectories corresponding to $E = 2\omega^2$ separate the two types of motion, oscillatory and rotary, for which reason these trajectories are called separatrices. Note that at $x = \pm(2j + 1)\pi\,(j = 0, 1, 2, \ldots)$ and $y = 0$ the points are singular points. A more general discussion will be given in the next subsection.

Stability Around a Singular Point

Equation (9.164) can be expressed in the general form of

$$\frac{dy}{dx} = \frac{p(x, y)}{y} \tag{9.167}$$

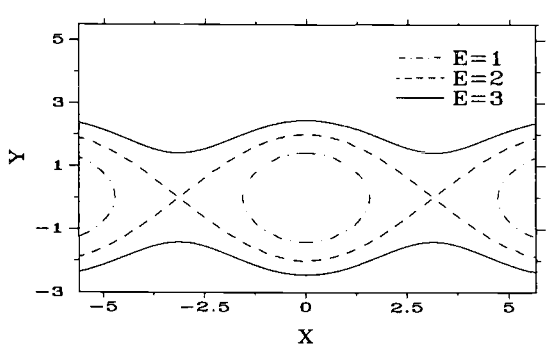

Fig. 9.22 Three trajectories.

The singular points of the equation are specified by

$$p(x, y) = y = 0 \tag{9.168}$$

Equation (9.167) is actually combined from the following two equations:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = p(x, y) \tag{9.169}$$

Let us construct a new set of coordinates u, v parallel to x, y with the origin at the singular point x_s and y_s, i.e.,

$$x = x_s + u, \qquad y = y_s + v$$

Because x_s and y_s are definite constants

$$\frac{dy}{dx} = \frac{dv}{du} \tag{9.170}$$

Expanding $p(x, y)$ into the Taylor series about the singular point (x_s, y_s), we obtain for $p(x, y)$

$$p(x, y) = p(x_s, y_s) + \left(\frac{\partial p}{\partial u}\right)_s u + \left(\frac{\partial p}{\partial v}\right)_s v$$

$$+ \frac{1}{2}\left(\frac{\partial^2 p}{\partial u^2}\right)_s u^2 + \cdots = cu + ev \tag{9.171}$$

Then Eq. (9.167) becomes

$$\frac{dv}{du} = \frac{cu + cv}{v}$$

or

$$\frac{du}{dt} = v$$

$$\frac{dv}{dt} = cu + ev$$

which can be rewritten in the matrix form as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c & e \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{9.172}$$

With the use of modal matrix discussed in Section 9.2, the preceding equation can be transformed into the equation for principal mode:

$$\begin{pmatrix} u \\ v \end{pmatrix} = (P) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Then Eq. (9.170) becomes

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\xi = e^{\lambda_1 t}$$

$$\eta = e^{\lambda_2 t}$$

The solutions for u and v are

$$u = u_1 e^{\lambda_1 t} + u_2 e^{\lambda_2 t}$$

$$v = v_1 e^{\lambda_1 t} + v_2 e^{\lambda_2 t}$$

It is evident, then, that the stability of the system around the singular point depends on the eigenvalues λ_1 and λ_2 determined from the characteristic equation

$$\begin{vmatrix} -\lambda & 1 \\ c & (e - \lambda) \end{vmatrix} = 0$$

$$\lambda_{1,2} = \frac{e}{2} \mp \sqrt{\left(\frac{e}{2}\right)^2 + c}$$

Thus, if $(e/2)^2 + c < 0$, the motion is oscillatory; if $(e/2)^2 + c > 0$, the motion is aperiodic; if $e > 0$, the system is unstable; and if $e < 0$, the system is stable.

Example 9.19

Let us consider once again the pendulum of Example 9.18 governed by the differential equation

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 \sin x$$

Determine the stability around the singular points that have been found as

$$x = \pm j\pi \quad j = 0, 1, 2, \ldots$$

$$y = 0$$

Solution. Around $x = y = 0$,

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 \sin x$$

To use Eq. (9.167), we write

$$p(x, y) = -\omega^2 \sin x$$

When it is expanded around the singular point $x = y = 0$, we have

$$p(x, y) = -\omega^2 u$$

That means

$$\dot{u} = v, \qquad \dot{v} = -\omega^2 u$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The characteristic equation is simply

$$\begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 = -\omega^2$$

$$\lambda_{1,2} = \pm i\omega$$

Because the roots are pure, imaginary complex conjugate, we conclude that the motion in the neighborhood of the origin is stable.

Around the singular point $x = \pi$, $y = 0$,

$$p(x, y) = -\omega^2 \sin x$$

When it is expanded around the singular point $x = \pi$, $y = 0$, we have

$$p(x, y) = \omega^2 u$$

That means

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The characteristic equation for eigenvalues is

$$\begin{vmatrix} -\lambda & 1 \\ \omega^2 & -\lambda \end{vmatrix} = \lambda^2 - \omega^2 = 0$$

$$\lambda_{1,2} = \pm\omega$$

Because the roots are real but opposite in sign, the singular point is a saddle point. Clearly, the motion around $x = \pi$, $y = 0$ is unstable.

Problems

9.1. Two simple pendula of length s and bob mass m swing in a common vertical plane and are attached to two different support points. The masses are connected by a spring of constant k as shown in Fig. 4.4. The equations of motion are derived in Example 4.3 and are rewritten as follows:

$$ms^2\ddot{\theta}_1 + mgs\theta_1 + ks^2(\theta_1 - \theta_2) = 0$$

$$ms^2\ddot{\theta}_1 + mgs\theta_2 - ks^2(\theta_1 - \theta_2) = 0$$

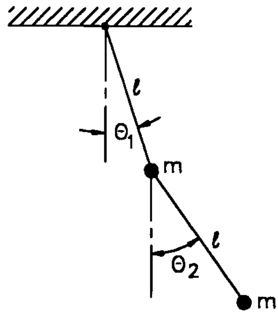

Fig. P9.2

Find the natural frequencies and the principal-mode solution for small oscillations of the system.

9.2. Determine the differential equations of motion for the double pendulum shown in Fig. P9.2. Find the natural frequencies and amplitude ratios for small oscillations of the system.

9.3. A two-degree-of-freedom system as shown in Fig. P9.3 is excited by a harmonic force $F_1 = F_0 \sin \omega_f t$. The physical constants for the system are $m_1 = 8$ kg, $m_2 = 4$ kg, $k_1 = 8.0$ kN/m, and $k_2 = 1.5$ kN/m. Using the Laplace transform method, determine the solution for the forced vibration with $F_0 = 2$N and $\omega_f = 2$ Hz. Assume that the initial displacements and velocities are zero.

9.4. Determine the solution of the vibrating system given in Example 9.7 with the use of the method of principal coordinates.

9.5. For a cantilevered beam with a uniform cross section, as shown in Fig. P9.5, find the transfer matrices from state 0 to 2. Determine the natural frequency of the system.

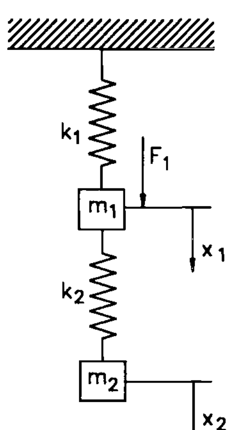

Fig. P9.3

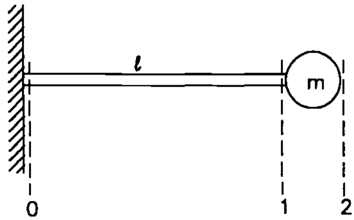

Fig. P9.5

9.6. A two-degree-of-freedom system consists of two equal springs and two equal masses as shown in Fig. P9.6. Using state vectors and transfer matrices, obtain the natural frequencies and mode shapes for the system.

9.7. Solve the problem of the vibrating string for the following boundary conditions: $y(0,t) = 0$; $y(L,t) = 0$; $\partial y/\partial t(x,0) = 0$; and $y(x,0) = f(x)$ as shown in Fig. P9.7.

9.8. A uniform string stretching from $-\infty$ to ∞ is originally displaced into the curve

$$y = \begin{cases} \sin x & 0 < x < \pi \\ 0 & \text{elsewhere} \end{cases}$$

Find the displacement of the string as a function of x and t.

9.9. Derive the differential equation of motion for a longitudinal vibration along a uniform rod with length L.

9.10. Consider a simply supported beam of length L. The initial displacement of the beam is

$$\frac{M_0}{2EI}x(L - X)$$

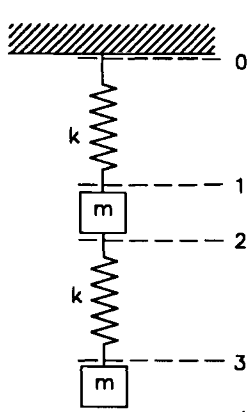

Fig. P9.6

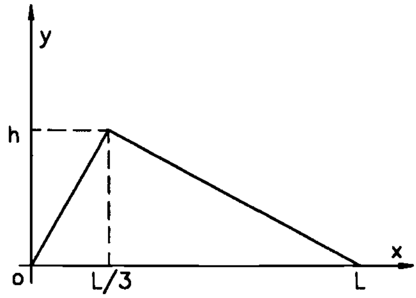

Fig. P9.7

and it is released from rest. Obtain the transverse motion $y(x, t)$ of the beam.

9.11. For a freely vibrating square membrane of length L, supported along the boundary $x = 0, x = L, y = 0, y = L$, suppose that the membrane is deflected in the form

$$w(x, y, 0) = f(x, y)$$

and is released from rest. Prove that the expression for the transverse vibration $w(x, y, t)$ of the membrane is

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{L} \cos \omega_{mn} t$$

where

$$a_{mn} = \frac{4}{L^2} \int_0^L \int_0^L f(x, y) \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{L} \mathrm{d}x\, \mathrm{d}y$$

9.12. The differential equation of motion of a damped pendulum can be written in the form

$$\ddot{\theta} + 2\zeta\omega\dot{\theta} + \omega^2 \sin\theta = 0$$

(a) Transform the equation into the equation for the phase plane and determine the singular points.

(b) Choose a value of ω, and plot curves in the phase plane for two cases: $\zeta = 0.1$, and $\zeta = 2$.

(c) Examine the motion in the neighborhood of the singular points.

9.13. Using the Runge–Kutta method, obtain the numerical solution $\theta(t)$ for

$$\ddot{\theta} + w^2 \sin\theta = 0$$

with $\omega^2 = 50$ (rad/s^2), $\theta(0) = \pi/2$, and $\dot{\theta}(0) = 0.1$(rad/s). Plot the numerical results for $0 < t < 2$.

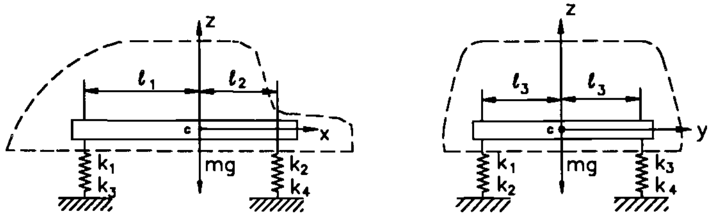

Fig. P9.14

9.14. Model the vibration of an automobile as a solid body supported by four springs as shown in Fig. P9.14. Obtain the differential equations of motion for the system under small oscillation.

9.15. Suppose that one of the four wheels is not balanced on the automobile modeled in Problem 9.14. Obtain the differential equations of motion for the system, and find the subsequent motion of the vibrating car during driving.

9.16. A circular membrane with radius of 10 cm is fixed on its edge. Suppose that the membrane is deflected initially in the form

$$w(r, 0) = \frac{10 - r}{10}$$

and is released from rest. Find the expression for the transverse vibration $w(r, t)$ of the membrane. Assume $a = 300$ m/s.

10
Special Relativity Theory

THIS chapter is devoted completely to the Special Relativity Theory. The reason behind this is to motivate readers to think, because, through this theory, space coordinates and time are related. Newton's equation of motion is modified, and times are different in moving and stationary systems. Furthermore, an event that occurs in the past in one system can become a future event in another system. All those are theoretically possible. To make these into our daily lives, further research is needed. Therefore, study of this theory not only can broaden our minds, but also can lead us to the invention of some new devices that will turn the theory into practical applications.

The development of this theory is based on famous experiments carried out by Michelson and Morley.* They found that the velocity of light always has the constant value despite the relative motions of source, observer, or medium. This result cannot be explained by the Galilean transformation that has been used throughout the previous chapters.

The set of transformations derived by Hendrik Antoon Lorentz, a Dutch physicist, solves that problem. The transformation is known as the Lorentz transformation and is the basis of the Special Relativity Theory. Albert Einstein in 1905 systematically recognized the limitations of Galilean transformation. He chose to modify the concept of time from absolute scale to space dependence. He made only two assumptions:

1) The laws of dynamics, including electromagnetic phenomena, must have the same form in systems moving with uniform velocity relative to each other.

2) The speed of light c is a universal constant, independent of any relative motion of the souce and the observer.

Using these assumptions, Einstein was able to formulate logically precise theories. The Special Relativity Theory of 1905 considers reference systems that are in uniform motion with respect to one another. The more general treatment of accelerated reference systems is the subject of the General Relativity Theory that was developed in 1915.

In Section 10.1, we shall discuss the Lorentz transformation. The conditions and assumptions, which are made for this transformation, are discussed in detail. In Section 10.2, we shall study the Brehme diagram, which is a graphical representation of the Lorentz transformation. The construction and the interpretation of this diagram will be discussed in the section. Through the example of the Brehme diagram, we can see that a region for past events in a system can be a region for future events in another system. Lastly, some consequences of the Special Relativity Theory are presented in Section 10.3. We shall discuss how to change some equations of Newtonian mechanics into relativistic forms.

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*For the Michelson–Morley experiment, see Silberstein, L., *The Theory of Relativity*, Macmillan, London, 1924, p. 71.

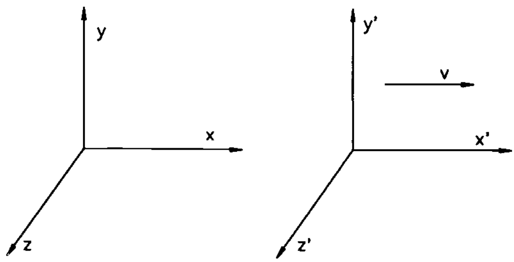


Fig. 10.1 Primed system moving with velocity v relative to the unprimed system.

10.1 Lorentz Transformation

Consider two reference systems as shown in Fig. 10.1. The primed system is moving with uniform velocity v along the x axis relative to the unprimed system. The Michelson–Morley experiments may be described as follows. A spark from a light source is emitted from the common origin of the two systems when they are coincided. As the light wave propagates spherically into the space, it is found that the spherical wave is the same in both systems regardless where the spark is released in the moving system or the stationary system. It is also found that the spherical wave front is not affected by whether or not the medium was moving.

This situation cannot be explained by the Galilean transformation that can be written as

$$x = x' + vt \qquad y = y' \qquad z = z' \qquad t = t$$

Under this transformation, the spherical wave in one system will be distorted in the other system because of the vt term in the x direction. However, if the second assumption of Einstein as stated above is observed, the equations for spherical surfaces in the both systems can be written as

$$x^2 + y^2 + z^2 = (ct)^2 \tag{10.1}$$

$$x'^2 + y'^2 + z'^2 = (ct')^2 \tag{10.2}$$

To derive the Lorentz transform, we define

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z, \qquad x_4 = ict$$

$$x_1' = x', \qquad x_2' = y', \qquad x_3' = z', \qquad x_4' = ict'$$

where $i = \sqrt{-1}$. This is known as Minkowski space, which is the complex four-dimensional space time. To establish the relationship between the two systems, we assume

$$x_\alpha' = \sum_{\beta=1}^{4} a_{\alpha\beta} x_\beta \tag{10.3}$$

and four unit vectors are orthogonal to each other. Using matrix notation, Eq. (10.3) becomes

$$X' = AX$$

where

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{14} & 0 & 0 & a_{44} \end{pmatrix}$$

Using the orthogonality assumption, we have

$$AA^T = I$$

i.e.,

$$a_{11}^2 + a_{14}^2 = 1, \qquad a_{41}^2 + a_{44}^2 = 1, \qquad a_{11}a_{41} + a_{14}a_{44} = 0 \tag{10.4}$$

Here we have three equations, but there are four unknowns to be determined. One additional equation is obtained from the relationship between the origin of the primed system and the corresponding coordinate in the unprimed system:

$$x_1 = vt = -\frac{iv}{c}(ict) = -i\beta x_4 \tag{10.5}$$

where $\beta = v/c$. But for this origin, we also can write

$$x_1' = 0 = a_{11}x_1 + a_{14}x_4 = (-a_{11}i\beta + a_{14})x_4$$

Hence we have

$$a_{14} = i\beta a_{11} \tag{10.6}$$

Using the preceding equation together with the three equations in Eq. (10.4) from orthogonality, we find

$$a_{11} = \frac{1}{\sqrt{1-\beta^2}} = \gamma \tag{10.7a}$$

i.e.,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$a_{14} = i\beta\gamma \tag{10.7b}$$

$$a_{44} = \gamma \tag{10.7c}$$

$$a_{41} = -i\beta\gamma \tag{10.7d}$$

Therefore the Lorentz matrix is

$$A = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{10.8}$$

The transformation can then be written explicitly as

$$x_1' = \gamma(x_1 + i\beta x_4)$$

or

$$x' = \gamma(x - vt) \tag{10.9a}$$

$$y' = y \tag{10.9b}$$

$$z' = z \tag{10.9c}$$

$$x_4' = \gamma(-i\beta x_1 + x_4)$$

or

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \tag{10.9d}$$

Not that as $v \ll c$, $\beta \to 0$, and $\gamma = 1$, the Lorentz transformation reduces to the Galilean transformation.

Because the two systems are in relative motion, the unprimed system may be considered as moving with velocity $-v$ along x' axis. The relations between the two systems can be written as

$$x = \gamma(x' + vt) \tag{10.10a}$$

$$y = y' \tag{10.10b}$$

$$z = z' \tag{10.10c}$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right) \tag{10.10d}$$

Applications of the Lorentz Transformation

Length contraction of a rigid rod. Consider a rigid rod of length ℓ

$$\ell = x_2 - x_1$$

along the unprimed x axis. When this rod is carried in the moving system, we find

$$\ell = x_2 - x_1 = \gamma(x_2' + vt') - \gamma(x_1' + vt') = \gamma(x_2' - x_1')$$

or

$$x_2' - x_1' = \ell' = \sqrt{1 - \beta^2}\,\ell \tag{10.11}$$

This means that if the rigid rod is measured in the moving system, the length becomes shorter because

$$\sqrt{1 - \beta^2} < 1$$

This fact is known as the Lorentz–Fitzgerald contraction.

Moving clock. Consider a clock fixed at the origin of the moving system

$$x' = 0, \qquad x = vt$$

The conversion of time gives

$$\begin{aligned} t_2' - t_1' &= \gamma\left[\left(t_2 - \frac{v}{c^2}x_2\right) - \left(t_1 - \frac{v}{c^2}x_1\right)\right] \\ &= \gamma\left[\left(t_2 - \frac{v^2}{c^2}t_2\right) - \left(t_1 - \frac{v^2}{c^2}t_1\right)\right] \\ &= \gamma(1 - \beta^2)(t_2 - t_1) \\ &= \sqrt{1 - \beta^2}(t_2 - t_1) \end{aligned} \tag{10.12}$$

That means that the time in the moving system becomes shorter than the time in the stationary system. Just to see the dramatic effect of the result, let us consider $\beta^2 = 0.99$. We find one year in the moving system; the corresponding time in the stationary system is 10 years.

Verification of Lorentz Transformation with Light Pulse Released from Different Systems

A light pulse released at the origin of the unprimed system at the instant when the two origins are coincident. The wave front in the positive x direction is at

$$x = ct$$

Using Eq. (10.9a) and Eq. (10.10d), the corresponding position in the x' system is determined as follows:

$$\begin{aligned} x' &= \gamma(x - vt) = \gamma(ct - vt) = \gamma(c - v)t \\ &= \gamma(c - v)\left[\gamma\left(t' + \frac{v}{c^2}x'\right)\right] \\ &= \gamma^2\left[(c - v)t' + \frac{v}{c}x' - \frac{v^2}{c^2}x'\right] \end{aligned}$$

Simplifying leads to

$$x' = ct' \tag{10.13}$$

The wave front in the negative x direction is at

$$x = -ct$$

The corresponding position in the moving system is

$$\begin{aligned} x' &= \gamma(x - vt) = \gamma(-ct - vt) \\ &= -\gamma(c + v)t = -\gamma(c + v)\left[\gamma\left(t' + \frac{v}{c^2}x'\right)\right] \end{aligned}$$

Rearranging gives

$$x' = -ct' \tag{10.14}$$

Therefore, they are spherical surfaces in both systems.

A light pulse is released at the origin of the primed system moving with velocity v at the instant when the two origins are coincident.

The wave front in the positive x' direction is

$$x' = ct'$$

The corresponding position of the wave front in the unprimed system is

$$\begin{aligned} x &= \gamma(x' + vt') = \gamma(ct' + vt') \\ &= \gamma(c + v)\left[\gamma\left(t - \frac{v}{c^2}x\right)\right] \end{aligned}$$

Simplifying leads to

$$x = ct$$

The wave front in the negative x' direction is

$$x' = -ct'$$

The corresponding wave front in the negative x direction is

$$\begin{aligned} x &= \gamma(x' + ct') = \gamma(-ct' + vt') = -\gamma(c - v)t' \\ &= -\gamma(c - v)\left[\gamma\left(t - \frac{v}{c^2}x\right)\right] \end{aligned}$$

Rearranging gives

$$x = -ct$$

Therefore, we find that the wave fronts are spherical surfaces in both systems.

10.2 Brehme Diagram

The Brehme diagram is a very useful tool for visualizing the Lorentz transformation. Note that in four-dimensional space time, y and z are not changed, but x and t are transformed into x' and t'. Therefore, the Brehme diagram is designed to show the relationship of Lorentz transformation and the coordinates of an event in the both systems, moving and stationary.

First let us construct the Brehme diagram as follows:

1) From the known value of velocity v of the moving system, calculate α such that

$$\alpha = \sin^{-1}\frac{v}{c} \tag{10.15}$$

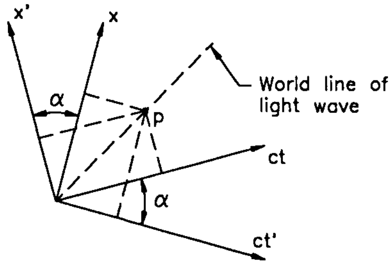


Fig. 10.2 Construction of the Brehme diagram.

2) Draw two straight axes for ct' and ct, with ct axis rotated by the angle α counterclockwise as shown in Fig. 10.2.

3) Draw x axis perpendicular to ct' axis.

4) Draw x' axis perpendicular to ct axis. Note that x, ct axes are for the unprimed system and x', ct' axes for primed system.

5) Draw a line from the origin bisecting the angle between axes of either primed or unprimed system. This line is called a world line of the light pulse. Any point P on this line, as shown in Fig. 10.2, represents the same spherical wave front in the two systems.

To see the significance of the Brehme diagram, let us take any point A as shown in Fig. 10.3.

$$\sin\alpha = \beta$$

$$\theta = 90 \deg - \alpha$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\cos\alpha}$$
$$= \sec\alpha$$

At point A

$$x = OT + PT = OD\sec\alpha + PA\tan\alpha$$
$$= x'\gamma + ct'\gamma\beta = \gamma(x' + c\beta t') = \gamma(x' + vt') \qquad (10.16)$$

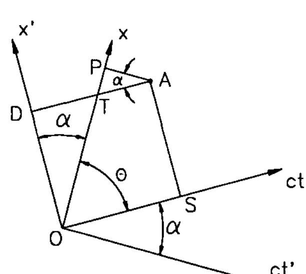


Fig. 10.3 Verification of the Brehme diagram.

This agrees with Eq. (10.10a). Also at point A

$$ct = OS = DT + TA = OD\tan\alpha + PA\sec\alpha$$
$$= x'\gamma\beta + ct'\gamma = \gamma(ct' + \beta x')$$

or

$$t = \gamma[t' + (v/c^2)x'] \tag{10.17}$$

which agrees with Eq. (10.10d). Therefore, the Brehme diagram truly reproduces the features of the Lorentz transformation.

Example 10.1

Suppose that one system is moving with a constant velocity of 2.598×10^8 m/s relative to another system. Choose the x and x' axes along the direction of the velocity. 1) Construct a Brehme diagram to relate the both systems. 2) Using the Brehme diagram constructed in step 1, indicate the regions in the diagram that represent the future in one system but the past in the other system. 3) Determine also the regions that represent the left of the reference position in one system but the right of the reference position in the other system.

Solution. 1) Construction of a Brehme diagram

$$\sin\alpha = \beta = \frac{v}{c} = \frac{2.598}{3.000} = 0.866$$
$$\alpha = 60 \text{ deg}$$

The Brehme diagram is constructed as shown in Fig. 10.4.

2) Take point P as a reference point. Draw a line PR from P perpendicular to ct axis. Note that the region to the left of line PR represents events taking

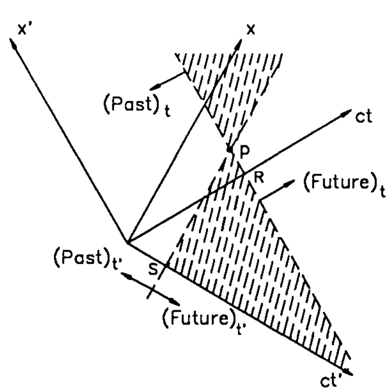


Fig. 10.4 Future and past overlapping in the shaded regions.

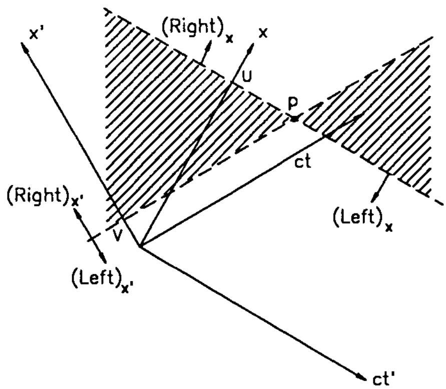


Fig. 10.5 Right and left overlapping in the shaded regions.

place in the past in the unprimed system; to the right of that line represents events that are to take place in the future. On the other hand, draw a line PS from P perpendicular to ct' axis. The region to the left of line PS represents events that have occurred in the past, and the region to the right of line PS represents events that are going to happen in the future in the primed system. The shaded regions represent the future in one system but the past in the other system as shown in Fig. 10.4.

3) From P, draw a line PU, normal to the x axis as shown in Fig. 10.5. The region below line PU means that events happen at places with small values in x coordinate as denoted by "Left"; above line PU represents events taking place at larger values of x as indicated by "Right." On the other hand, the line PV is normal to x' axis. The region below line PV represents events happen at places with smaller value in x', and the region above line PV represents that events take place at the larger value of x'. The shaded regions depict "Right" in one system and "Left" in the other system.

Example 10.2

Consider a case of twin brothers A and B. B takes a space trip traveling with relativistic velocity to another planet. After arriving on the planet, B stays briefly and then comes back. Assume that the time for acceleration in the beginning of the trip, brief stay on the planet, and for the deceleration at the end of return trip are small compared to the duration of whole trip. Determine which brother, A or B, is actually younger in age.

Solution. Construct a Brehme diagram for the two systems as shown in Fig. 10.6a for the trip of B to the planet. Consider x', ct' are the axes for the moving system and x and ct axes for the stationary system. Assume that A, B are always at the origins of space coordinates. The time spent by B is Δt_b and by A is Δt_a. From the diagram we find

$$\Delta t_b = \Delta t_a \cos\alpha$$

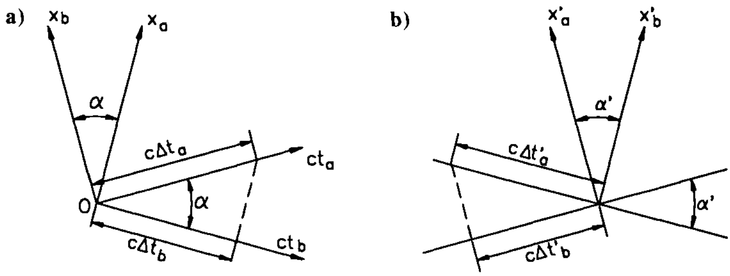


Fig. 10.6 Brehme diagrams for brother B.

Construct another Brehme diagram for the two systems as shown in Fig. 10.6b for the return trip of B. Again assume that A, B are at the origins of space coordinates. Converting the time, we have

$$\Delta t'_b = \gamma\left(t_2 + \frac{v}{c^2}x_2\right)_a - \gamma\left(t_1 + \frac{v}{c^2}x_1\right)_a = \gamma(t_2 - t_1)_a + \gamma\frac{v}{c^2}(x_2 - x_1)_a$$

$$= \gamma(t_2 - t_1)_a - \gamma\frac{v^2}{c^2}(t_2 - t_1)_a = \Delta t'_a \cos\alpha'$$

The prime symbol is used only to indicate the quantities of the return trip. In this way the expression includes the possibility that α' may be different from α. The total time for the round trip of B is

$$\Delta t_b + \Delta t'_b = \Delta t_a \cos\alpha + \Delta t'_a \cos\alpha'$$

Therefore, we conclude that B is younger.

10.3 Immediate Consequences in Kinematics and Dynamics

Addition of Velocities

Suppose that point P in the moving system moves with velocity u along x' axis. The velocity of the moving system is v. The velocity of P in the stationary system is no longer $u + v$ under the Lorentz transformation because

$$x' = \gamma(x - vt) \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$u = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - v\,dx/c^2} = \frac{dx/dt - v}{1 - (v/c^2)(dx/dt)}$$

Solving for the velocity of P in the stationary system, we find

$$\frac{dx}{dt} = \frac{u + v}{1 + uv/c^2} \tag{10.18}$$

Note that if u and v are small compared with c, then the velocity of P reduces to the addition of u and v. We can apply this result to the superposition of two Lorentz transformations. Consider three frames of reference S, S^*, and S^{**}. S^* has the velocity v relative to S, and S^{**} has the velocity u relative to S^*. The transformation equations relating S^{**} and S are

$$x^{**} = \gamma_w(x - wt) \tag{10.19a}$$

$$y^{**} = y \tag{10.19b}$$

$$z^{**} = z \tag{10.19c}$$

$$t^{**} = \gamma_w\left(t - \frac{wx}{c^2}\right) \tag{10.19d}$$

where

$$w = \frac{u + v}{1 + uv/c^2} \tag{10.19e}$$

and

$$\gamma_w = \frac{1}{\sqrt{1 - (w/c)^2}} \tag{10.19f}$$

Equations of Motion in Relativistic Form

Time change. Consider a particle at rest in the primed system. The velocity of the particle in the unprimed system is v. The changes of time in the two systems are related by

$$(\mathrm{d}\tau)^2 = (\mathrm{d}t)^2\left[1 - \frac{v^2}{c^2}\right] = (\mathrm{d}t)^2(1 - \beta^2)$$

or

$$\mathrm{d}\tau = \mathrm{d}t\sqrt{1 - \beta^2} \tag{10.20}$$

where $\mathrm{d}\tau$ is the change of time in the moving system and $\mathrm{d}t$ is the change of time in the stationary system.

Equation of motion. By defining the relativistic mass and momentum as

$$m \equiv \frac{m_0}{\sqrt{1 - \beta^2}} = \gamma m_0 \qquad P_i \equiv m_0\frac{\mathrm{d}x_i}{\mathrm{d}t} \tag{10.21}$$

and the four-component force as

$$\mathcal{F}_i \equiv m_0\frac{\mathrm{d}^2x_i}{\mathrm{d}\tau^2} \tag{10.22}$$

we find that the Newtonian equation becomes

$$\mathcal{F}_i = m_0 \frac{d^2 x_i}{d\tau^2} = \frac{d}{d\tau} m_0 \frac{dx_i}{d\tau} = \frac{d}{d\tau} m_0 \gamma \frac{dx_i}{dt} = \gamma \frac{d}{dt} m_0 \gamma v_i = \gamma \frac{d}{dt} m v_i$$

According to the meaning of the term, we obtain

$$\frac{d}{dt} m v_i = \mathcal{F}_i / \gamma = F_i \tag{10.23}$$

i.e.,

$$\frac{d}{dt} \frac{m_0 v_i}{\sqrt{1-\beta^2}} = F_i$$

where m_0 is the rest mass, $\mathcal{F}_i$ is the ith component of the Minkowski force, and F_i is the ith component of the coordinate force as observed in the unprimed system.

Relativistic energy. By defining four vector components

$$\mu_1 \equiv \frac{dx}{d\tau}, \qquad \mu_2 \equiv \frac{dy}{d\tau}, \qquad \mu_3 \equiv \frac{dz}{d\tau}, \qquad \mu_4 \equiv ic \frac{dt}{d\tau} = ic\gamma$$

then we have

$$\begin{aligned} \sum_{i=1}^{4} \mu_i^2 &= -c^2 \left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 \\ &= \left(\frac{dt}{d\tau}\right)^2 (-c^2 + v^2) = \frac{1}{1-\beta^2}(v^2 - c^2) = -c^2 \end{aligned} \tag{10.24}$$

Note that μ_i is the component of the proper velocity that is obtained by the distance traveled in the unprimed system divided by time interval from moving clock; v_i is the component of the coordinate velocity that is the distance divided by the time interval from a fixed clock. On the other hand, taking the dot product of the Minkowski force with the proper velocity gives

$$\begin{aligned} \sum_{i=1}^{4} \mathcal{F}_i \mu_i &= m_0 \sum_i \frac{d\mu_i}{d\tau} \mu_i = \frac{1}{2} m_0 \frac{d}{d\tau} \sum_{i=1}^{4} \mu_i^2 \\ &= \frac{1}{2} m_0 \frac{d}{d\tau}(-c^2) = 0 \end{aligned}$$

$$\mathcal{F}_4 = \frac{-1}{\mu_4} \sum_{i=1}^{3} \mathcal{F}_i \mu_i = \frac{-1}{\mu_4} \sum_{i=1}^{3} \gamma F_i \gamma v_i = \frac{i\gamma}{c} (\boldsymbol{F} \cdot \boldsymbol{V})$$

or

$$\frac{i\gamma}{c}(\boldsymbol{F}\cdot\boldsymbol{V}) = \mathcal{F}_4 = m_0\frac{\mathrm{d}}{\mathrm{d}\tau}\mu_4 = m_0\frac{\mathrm{d}}{\mathrm{d}\tau}(ic\gamma)$$

$$(\boldsymbol{F}\cdot\boldsymbol{V}) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{m_0c^2}{\sqrt{1-\beta^2}}$$

But the meaning of $\boldsymbol{F}\cdot\boldsymbol{V}$ is the rate change of the kinetic energy as discussed in Section 2.3; therefore, we find

$$\text{K.E.} = \frac{m_0c^2}{\sqrt{1-\beta^2}} \tag{10.25}$$

Note that Eqs. (10.23) and (10.25) are usually given in college physics books without explanation.

Example 10.3

Consider that a particle is moving on the x axis under a constant coordinate force F. Assume it starts from rest at $t = 0$. Find the maximum limiting velocity of the particle.

Solution. Using Eq. (10.23), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0v}{\sqrt{1-v^2/c^2}}\right) = F$$

Integrating leads to

$$\frac{m_0v}{\sqrt{1-v^2/c^2}} = Ft$$

Solving for v, we find

$$v = \frac{cFt}{\sqrt{(Ft)^2+(m_0c)^2}}$$

Therefore, the velocity of the particle is always less than c. The limiting value is c as t approaches infinity. This is not true in Newtonian mechanics.

Problems

10.1. Show that the wave equation

$$\nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

is invariant under a Lorentz transformation but not under a Galilean transformation.

10.2. Derive the relationship for the velocity of a particle along the x' axis in a primed system to the velocity in the unprimed system under the Lorentz transformation.

10.3. Derive the relationship for the acceleration of a particle along x' axis in the primed system compared to that in the unprimed system under the Lorentz transformation.

10.4. Do the following:

(a) Determine the velocity of a moving system such that 7 days in the moving system is equivalent to 1 year in the stationary system.

(b) Construct a Brehme diagram with scales on the axes for the systems determined in part (a).

10.5. Verify the Brehme diagram Fig. 10.6b for the return trip of B.

10.6. Do the following:

(a) Determine the velocity of a moving system so that an observer in the moving system can see an event that happened one day ago in the stationary system.

(b) Construct a Brehme diagram and mark the position of the observer in the diagram for the systems described in part (a).

10.7. Prove that if velocities u and v are less than speed of light c, the result of the addition of velocities through relativity theory can never be greater than c.

10.8. Prove that the kinetic energy expressed by Eq. (10.25) will reduce to K.E. $= \frac{1}{2}m_0v^2 + m_0c^2$ if $v \ll c$. Discuss the significance of m_0c^2.

10.9. Consider a system moving along x axis with velocity v relative to the stationary system. A sphere in the moving system is described by

$$x'^2 + y'^2 + z'^2 = a^2$$

What will be the shape as observed in the stationary system?

10.10. Two particles, with rest masses m_1, m_2, move along the x axis with velocities u_1, u_2, respectively. They collide and coalesce to form a single particle. Assuming the laws of conservation of relativistic mass and momentum, prove that the rest mass m_3 and velocity u_3 of the resulting single particle are given by

$$m_3^2 = m_1^2 + m_2^2 + 2m_1m_2\,\gamma_1\gamma_2\left(1 - \frac{u_1u_2}{c^2}\right)$$

$$u_3 = \frac{m_1\gamma_1u_1 + m_2\gamma_2u_2}{m_1\gamma_1 + m_2\gamma_2}$$

where

$$\gamma_1 = \frac{1}{\sqrt{1-(u_1/c)^2}}, \qquad \gamma_2 = \frac{1}{\sqrt{1-(u_2/c)^2}}$$

Appendix A: Runge–Kutta Method

THE Runge–Kutta computation scheme introduced here is accurate to the fourth-order-of-time increment. The equations to be integrated may be written as

$$\frac{dx}{dt} = f(t, x, y) \tag{A.1a}$$

$$\frac{dy}{dt} = g(t, x, y) \tag{A.1b}$$

where $f(t, x, y)$ and $g(t, x, y)$ are known functions. The values of x and y are known at $t = t_i$. Hence at $t = t_i = T_1$,

$$x_1 = x_i, \qquad y_1 = y_i$$

$$F_1 = f(T_1, x_1, y_1), \qquad G_1 = g(T_1, x_1, y_1)$$

At $t = t_i + h/2 = T_2$, where $h = \Delta t$,

$$x_2 = x_i + F_1 h/2, \qquad y_2 = y_i + G_1 h/2$$

$$F_2 = f(T_2, x_2, y_2), \qquad G_2 = g(T_2, x_2, y_2)$$

At $t = t_i + h/2 = T_3$,

$$x_3 = x_i + F_2 h/2, \qquad y_3 = y_i + G_2 h/2$$

$$F_3 = f(T_3, x_3, y_3), \qquad G_3 = g(T_3, x_3, y_3)$$

At $t = t_i + h = T_4$,

$$x_4 = x_i + F_3 h, \qquad y_4 = y_i + G_3 h$$

$$F_4 = f(T_4, x_4, y_4), \qquad G_4 = g(T_4, x_4, y_4)$$

Then the values of x and y for next step $t = t_i + \Delta h$ are

$$x_{i+1} = x_i + (h/6)[F_1 + 2F_2 + 2F_3 + F_4] \tag{A.2a}$$

$$y_{i+1} = y_i + (h/6)[G_1 + 2G_2 + 2G_3 + G_4] \tag{A.2b}$$

Note that the formulas that have been given can be applied repeatedly until the expected time is reached. To save time of computation, a large value of h may be chosen, but the accuracy of the calculation may suffer. On the other hand, for a very small value of h, the number of steps must increase in order to reach the final value of time. For each step of calculation, the computer will create some error caused by the limitation of the number of digits calculated in the computer. Therefore, a compromised value of h must be chosen for actual applications. The way to determine the value of h may be done as follows: choose a value for h, say h_1, and complete the calculation for a set of x and y. Reduce the value of h to $h_1/10$, for example, and repeat the calculation. Compare the new result with the old set. If the results are not significantly changed, then the value of h chosen initially is good enough for calculation.

The preceding formulation can be used for solving a second-order differential equation. For example,

$$\ddot{x} = (1/m)[F(t) - kx - c\dot{x}] \tag{A.3}$$

we can define

$$\dot{x} = y \tag{A.4a}$$

then

$$\dot{y} = (1/m)[F(t) - kx - cy] = g(t, x, y) \tag{A.4b}$$

Equations (A.4a) and (A.4b) are in the form of Eqs. (A.1a) and (A.1b). Therefore, the method can be applied. In fact, the method can be extended to integrate many equations simultaneously.

Appendix B: Stoke's Theorem

B.1 Proof of Green's Lemma in *XY* Plane

Green's lemma can be written in equational form as

$$\int_C (P\,\mathrm{d}x + Q\mathrm{d}y) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial p}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y \tag{B.1}$$

Consider first the integral

$$I_1 = \iint_R \frac{\partial Q}{\partial x}\mathrm{d}x\,\mathrm{d}y$$

Let the curve $P_1P_4P_3$ be represented by

$$x = g_1(y)$$

and $P_1P_2P_3$ by

$$x = g_2(y)$$

as shown in Fig. B.1. Then we have

$$I_1 = \int_c^d \int_{g_1(y)}^{g_2(y)} \frac{\partial Q}{\partial x}\mathrm{d}x\,\mathrm{d}y = \int_c^d \{Q[g_2(y), y] - Q[g_1(y), y]\}\mathrm{d}y$$

$$= \int_c^d Q[g_2(y), y]\mathrm{d}y + \int_d^c Q[g_1(y), y]\mathrm{d}y$$

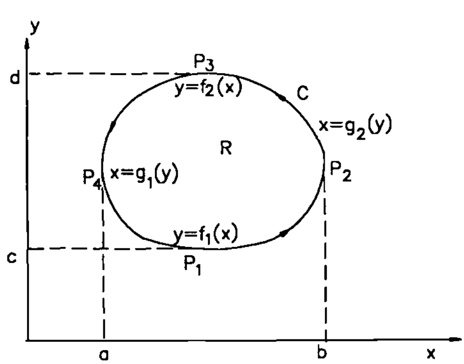


Fig. B.1 Green's lemma.

Because the first part represents the integral for the curve $P_1P_2P_3$ and the second part for the curve $P_3P_4P_1$, the result is

$$I_1 = \int_C Q(x, y)\mathrm{d}y$$

Similarly, denoting the curve $P_4P_1P_2$ by $y = f_1(x)$ and the curve $P_2P_3P_4$ by $y = f_2(x)$, we have

$$\begin{aligned}
I_2 &= \iint_R \frac{\partial P}{\partial y}\mathrm{d}x\,\mathrm{d}y = \iint_R \frac{\partial P}{\partial y}\mathrm{d}y\ \mathrm{d}x \\
&= \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y}\mathrm{d}y\mathrm{d}x = \int_a^b \{P[x, f_2(x)] - P[x, f_1(x)]\}\mathrm{d}x \\
&= -\int_b^a P[x, f_2(x)]\mathrm{d}x - \int_a^b P[x, f_1(x)]\mathrm{d}x
\end{aligned}$$

Because the first part represents the integral along $P_2P_3P_4$ and the second part for the curve $P_4P_1P_2$, the combined result is

$$I_2 = -\int_C P\,\mathrm{d}x$$

Therefore,

$$I_1 - I_2 = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = \int_C (p\ \mathrm{d}x + Q\ \mathrm{d}y)$$

Green's lemma is established.

B.2 Stoke's Theorem

First consider a two-dimensional surface. Let

$$\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j}$$

$$\boldsymbol{R} = x\boldsymbol{i} + y\boldsymbol{j}$$

$$\mathrm{d}\boldsymbol{R} = \boldsymbol{i}\mathrm{d}x + \boldsymbol{j}\mathrm{d}y$$

then

$$P\ \mathrm{d}x + Q\ \mathrm{d}y = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{R}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \begin{vmatrix} 0 & 0 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \boldsymbol{k}\cdot\nabla\times\boldsymbol{F}$$

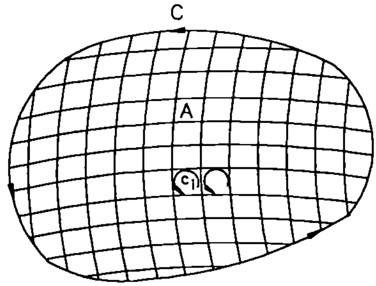


Fig. B.2 Stoke's theorem applies to a three-dimensional case.

Therefore, Green's lemma becomes

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R} = \iint_A \boldsymbol{k} \cdot \nabla \times \boldsymbol{F} \mathrm{d}A = \iint_S \nabla \times \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} \tag{B.2}$$

where $\mathrm{d}\boldsymbol{S} = \boldsymbol{k}\mathrm{d}A$.

Equation (B.2) can be generalized to three-dimensional surface bounded by curve C. We divide the surface into many infinitesimal areas as shown in Fig. B.2. Each area can be treated as a two-dimensional surface. The sum of all the area integrals is the integral for the three-dimensional surface, and the sum of all the line integrals along C_i is the line integral along C because the line integrals in the interior areas are cancelled by each other. Therefore Eq. (B.2) can be applied to a three-dimensional case.

Table C.1 Planetary data[a]

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
μ,[b] km^3/s^2	2.232×10^4	3.257×10^5	3.986×10^5	4.305×10^4	1.268×10^8	3.795×10^7	5.820×10^6	6.896×10^6	3.587×10^5
Equatorial diameter, km	4670	12,400	12,700	6760	143,000	121,000	47,100	50,700	5950
Mean distance from sun 10^6 km	57.9	108	149.6	227.7	777.8	1486	2869	4475	5899
Sidereal period	88.0 days	224.7 days	365.3 days	687.0 days	11.86 yr	29.46 yr	84.01 yr	164.79 yr	248.43 yr
Axial rotation (equatorial)	58.7 days	243 days	23H 56M 04S	24H 56M 23S	9H 50M 30S	10H 14M	10H 49M	About 14H	6D 9H
Axial inclination, deg	?	?	23°27′	23°59′	3°04′	26°44′	97°53′	28°48′	?
Mean synodic period, days	115.9	584.0	—	779.9	398.9	378.1	369.7	367.5	366.7
Escape velocity, km/s	4.2	10	11	6.4	59.7	35.4	22.4	31	?
Density: water = 1	5.5	5.3	5.5	3.9	1.3	0.7	1.7	1.8	?
Volume: Earth = 1	0.06	0.86	1	0.15	1319	744	47	54	0.1?
Mass: Earth = 1	0.06	0.82	1	0.11	316	95	15	17	?
Surface gravity: Earth = 1	0.38	0.90	1	0.38	2.64	1.16	1.11	1.21	?
Max surface temperature, °F	+770	+887	+140	+80	−200	−240	−310	−360	?
Number of satellites	0	0	1	2	13	10	5	2	0

[a]By permission, *Pioneer to Jupiter*, Bendix Field Engineering Corp., Palo Alto, CA, Nov. 1974.
[b]μ = GM = Newtonian constant times planetary mass.

Appendix D: Determinants and Matrices

D.1 Definitions of Determinants and Matrices

A determinant of nth order is a set of n^2 numbers or symbols, which are called the elements, arranged between two vertical lines in the form of a square array of n rows and n columns. When it is expanded, the final result of a determinant is a single number or is essentially becoming a number at last.

A matrix is a rectangular array of numbers or symbols. Its size is specified by $m \times n$ where m is the number of rows and n the number of columns. In particular when n is 1, the matrix is called a column matrix and is equivalent to a vector in m dimensional space. Usually all the elements are arranged between two arcs. If $m = n$, then the matrix is called a square matrix. The elements in the matrix are not related to each other, and matrices cannot be expanded as a determinant. Therefore, $m \times n$ matrix means a set of $m \times n$ numbers arranged specifically according this form.

D.2 Properties of Determinants

1) Determinant of A equals determinant of A transposed:

$$|A| = |A^T| \tag{D.1}$$

2) If any two rows (or columns) are interchanged, the sign of the determinant is changed.

3) If each element of a row (or column) is multiplied by a constant k, the value of the determinant is multiplied by k:

$$\begin{vmatrix} ka_{11} & a_{12} \\ ka_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \tag{D.2}$$

4) The multiplication of determinants $|A|$ and $|B|$ equals the determinant of AB (product of matrices):

$$|A||B| = |AB| \tag{D.3}$$

5) If two rows (columns) of a determinant are identical or in proportion, the value of the determinant is zero.

6) A determinant of order of three can be expanded directly into the sum of six terms as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1 \tag{D.4}$$

7) The order of a determinant can be reduced by one if it is expanded as follows:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = (-1)^2 a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$+(-1)^3 a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + (-1)^4 a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \tag{D.5}$$

$$+(-1)^5 a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

The determinants behind elements a_{1i} are the determinants of minors. Note that through the preceding procedure the order of the determinant is reduced by one. When the order of determinant reaches three, the determinant can be expanded directly as given in Eq. (D.4).

In general,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} |\text{ minor of } a_{ij}| \tag{D.6}$$

D.3 Properties of Matrices

1) Addition of matrices.

$$A + B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{in} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

$$= (a_{ij} + b_{ij}) \tag{D.7}$$

2) Multiplication of two matrices.

$$C = AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2} \cdots \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2} \cdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + a_{m2}b_{22} + \cdots + a_{mn}b_{n2} \cdots \end{pmatrix}$$

$$(C_{ik}) = (a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}) \tag{D.8}$$

Note that in order to perform the multiplication of two matrices, if the order of A is $m \times n$, the order of B must be $n \times p$. The number n must be the same in A and B, i.e., the number of columns in A must be the same as the number of rows in B.

3) Multiplication of a matrix by a constant k.

$$kA = (ka_{ij}) \tag{D.9}$$

This means that all the elements must be multiplied by k.

4) Inverse of a matrix. If matrix A is a square matrix, the inverse of A may be obtained through the following procedure. Consider

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (a_{ij})$$

The minor of an element a_{ij} is M_{ij} that is the determinant of A except the ith row, jth column being omitted, and the cofactor is

$$(-1)^{i+j}M_{ij}$$

The adjoint of A is the cofactor transposed, i.e.,

$$\mathrm{adj}(A) = [(-1)^{i+j}M_{ij}]^T$$

The inverse of A is the adj(A) divided by $|A|$

$$A^{-1} = \frac{1}{|A|}\mathrm{adj}(A) \tag{D.10}$$

For example, let us construct the inverse of the following matrix

$$D = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 5 & 6 \\ 7 & 8 & -9 \end{pmatrix}$$

We find

$$((-1)^{i+j}M_{ij}) = \begin{pmatrix} -93 & 6 & -67 \\ 6 & -30 & -22 \\ -27 & -18 & -3 \end{pmatrix}$$

$$|D| = -306$$

Hence

$$D^{-1} = -\frac{1}{306}\begin{pmatrix} -93 & 6 & -27 \\ 6 & -30 & -18 \\ -67 & -22 & -3 \end{pmatrix}$$

Checking the result, we find $DD^{-1} = I$.

Appendix E: Method of Partial Fractions

To use the table of Laplace transforms, it is often necessary to employ the method of partial fractions. The method briefly can be outlined as follows: Suppose that the transformed function can be written as

$$F(s) = N(s)/D(s) \tag{E.1}$$

where $N(s)$ and $D(s)$ are polynomials. There are several ways to express $F(s)$ depending on the nature of roots of $D(s) = 0$. Note

$$\begin{aligned} D(s) &= s^n + c_1 s^{n-1} + c_2 s^{n-2} + \cdots + c_n \\ &= (s - a_1)(s - a_2)\cdots(s - a_n) \end{aligned} \tag{E.2}$$

1) Unrepeated factor of $(s - a_i)$. If the roots of $D(s)$ are different, the transformed function can be written as

$$F(s) = \frac{A_1}{s - a_1} + \frac{A_2}{s - a_2} + \cdots + \frac{A_n}{s - a_n} \tag{E.3}$$

To determine the constants A_i, we multiply both sides of the preceding equation by $D(s)$ and set s to a_i to find

$$A_i = \lim_{s \to a_i} (s - a_i) \frac{N(s)}{D(s)}$$

To simplify the expression, we rewrite the equation for A_i as

$$A_i = \lim_{s \to a_i} \frac{N(s)}{D_1(s)}$$

where

$$D_1(s) = \frac{D(s)}{s - a_i}$$

or

$$D(s) = (s - a_i) D_1(s)$$

Differentiating the equation gives

$$D'(s) = D_1(s) + (s - a_i) D_1'(s)$$

$$\lim_{s \to a_1} D'(s) = D_1(s)$$

Hence

$$A_i = \lim_{s \to a_i} \frac{N(s)}{D'(s)} \tag{E.4}$$

2) Repeated factor of $(s - a)^m$. For this case, the transformed function can be written as

$$F(s) = \frac{N(s)}{D(s)} = \frac{A_m}{(s-a)^m} + \frac{A_{m-1}}{(s-a)^{m-1}} + \cdots + \frac{A_1}{s-a} + w(s)$$

To determine the constants, we multiply the preceding equation by $(s - a)^m$ and find

$$\begin{aligned}(s-a)^m F(s) &= A_m + A_{m-1}(s-a) + \cdots + A_1(s-a)^{m-1} \\ &+ w(s)(s-a)^m = Q(s)\end{aligned} \tag{E.5}$$

Hence

$$A_m = \lim_{s \to a}[(s-a)^m F(s)] = Q(a)$$

Differenting Eq. (E.5) leads to

$$Q'(s) = A_{m-1} + \text{ terms containing factor } (s-a)$$

Hence

$$A_{m-1} = [Q'(s)]_{s=a}$$

Differentiating again, we find

$$[Q''(s)]_{s=a} = 2!\,A_{m-2}$$

Hence

$$A_{m-k} = \frac{1}{k!}\left[\frac{d^k Q(s)}{ds^k}\right]_{s=a} \tag{E.6}$$

3) Unrepeated complex factors $(s - a)$ $(s - \bar{a})$. When the roots of $D(s)$ are complex numbers, a and conjugate of $a\,(\bar{a})$, the function can be written as

$$F(s) = \frac{N(s)}{D(s)} = \frac{As + B}{(s-a)(s-\bar{a})} + w(s)$$

Let

$$a = \alpha + i\beta, \qquad \bar{a} = \alpha - i\beta$$

then

$$F(s) = \frac{As + B}{(s-\alpha)^2 + \beta^2} + w(s) \tag{E.7}$$

To determine the expression for $(As + B)$, we multiply the preceding equation by $(s - \alpha)^2 + \beta^2$, then we have

$$As + B + w(s)[(s-\alpha)^2 + \beta^2] = [(s-\alpha)^2 + \beta^2]F(s) = R(s)$$

Let s approach a, then

$$R(a) = S_a + iT_a = aA + B = (\alpha + i\beta)A + B$$

That means

$$S_a = \alpha A + B, \qquad T_a = \beta A$$

Hence

$$\begin{aligned} As + B &= A(s-\alpha) + \alpha A + B \\ &= \frac{T_a}{\beta}(s-\alpha) + S_a \end{aligned} \tag{E.8}$$

Example E.1

To determine

$$\mathcal{L}^{-1}\left\{\frac{s^2+1}{s^3+3s^2+2s}\right\}$$

Solution.

$$\mathcal{L}^{-1}\left\{\frac{s^2+1}{s^3+3s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{s^2+1}{s(s+1)(s+2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+2}\right\}$$

Note that

$$D'(s) = 3s^2 + 6s + 2$$

$$A_1 = \frac{N(0)}{D'(0)} = \frac{1}{2}$$

$$A_2 = \frac{N(-1)}{D'(-1)} = -2$$

$$A_3 = \frac{N(-2)}{D'(-2)} = \frac{5}{2}$$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{2s} - 2\frac{1}{s+1} + \frac{5}{2}\frac{1}{s+2}\right] = \frac{1}{2} - 2e^{-t} + \frac{5}{2}e^{-2t}$$

Example E.2

To determine

$$\mathcal{L}^{-1}\{(1/(s+1)(s^2+1)\}$$

Solution.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left[\frac{A}{s+1} + \frac{Bs+C}{s^2+1}\right]$$

$$N(s) = 1$$

$$D(s) = (s+1)(s^2+1), \qquad D'(s) = (s^2+1) + (s+1)(2s)$$

$$A = \frac{N(-1)}{D'(-1)} = \frac{1}{2}$$

$$R(s) = \frac{1}{s+1}$$

$$R(i) = \frac{1}{1+i} = \frac{1-i}{2} = iB + C$$

$$B = -\frac{1}{2}, \qquad C = \frac{1}{2}$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{A}{s+1} + \frac{Bs+C}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2(s+1)} + \frac{-S+1}{2(s^2+1)}\right\}$$

$$= \frac{1}{2}(e^{-t} + \sin t - \cos t)$$

Appendix F: Tables of Fourier and Laplace Transforms

Table F.1 Fourier transforms

Function	Transformed function
1. $f(t)$	$\mathcal{F}(u)$
2. $f(t-\tau)$	$\mathcal{F}(u)e^{-iu\tau}$
3. $f'(t)$	$iu\mathcal{F}(u)$
4. $f^{(n)}(t)$	$(iu)^n\mathcal{F}(u)$
5. $f(t)=h$ as $-a<t<a$ $=0$ elsewhere	$\frac{2h}{u}\sin ua$
6. $\delta(t)$	1
7. $\delta(t-\tau)$	$e^{-iu\tau}$
8. $H(t)$	$\frac{1}{iu}$
9. $H(t-\tau)$	$\frac{1}{iu}e^{-iu\tau}$
10. $H(t)e^{-\alpha t}$, $\alpha>0$	$\frac{1}{\alpha+iu}$
11. $H(t-\tau)e^{-\alpha t}$, $\alpha>0$	$\frac{1}{\alpha+iu}e^{-(\alpha+iu)\tau}$
12. $e^{-\alpha\|t\|}$, $\alpha>0$	$\frac{2\alpha}{\alpha^2+u^2}$
13. $H(t)t$	$\frac{-1}{u^2}$
14. $H(t)\sin at$	$\frac{a}{a^2-u^2}$
15. $H(t)\cos at$	$\frac{iu}{a^2-u^2}$

Table F.2 Laplace transforms

Function	Transformed function
1. $f(t)$	$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\mathrm{d}t$
2. $af(t) + bg(t)$	$aF(s) + bG(s)$
3. $f'(t)$	$sF(s) - f(0)$
4. $f''(t)$	$s^2F(s) - sf(0) - f'(0)$
5. $f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k} \dfrac{\mathrm{d}^{K-1}f(0)}{\mathrm{d}t^{k-1}}$
6. $\overbrace{\int_0^t \cdots \int_0^t}^{n\ \text{times}} f(t)\ \overbrace{\mathrm{d}t \ldots \mathrm{d}t}^{n\ \text{times}}$	$\dfrac{1}{s^n}F(s)$
7. $t^n f(t)$	$(-1)^n \dfrac{d^nF(s)}{\mathrm{d}s^n}$
8. $e^{at}f(t)$	$F(s-a)$
9. $\begin{cases} f(t-a) & \text{as } t > a \\ 0 & \text{as } t < a \end{cases}$	$e^{-as}F(s)$
10. $\int_0^t f(t-u)g(u)\mathrm{d}u$	$F(s)G(s)$
11. 1	$1/s$
12. e^{-at}	$\dfrac{1}{s+a}$
13. $\dfrac{1}{a-b}(e^{-bt} - e^{-at})$	$\dfrac{1}{(s+a)(s+b)}$
14. $\dfrac{1}{b-a}(be^{-bt} - ae^{-at})$	$\dfrac{s}{(s+a)(s+b)}$
15. $\sin at$	$\dfrac{a}{s^2+a^2}$
16. $\cos at$	$\dfrac{s}{s^2+a^2}$
17. $\sinh at$	$\dfrac{a}{s^2-a^2}$
18. $\cosh at$	$\dfrac{s}{s^2-a^2}$
19. $t \sin at$	$\dfrac{2as}{(s^2+a^2)^2}$
20. $t \cos at$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$

(Cont.)

Table F.2 Laplace transforms (continued)

Function	Transformed function
21. $\sin at - at\cos at$	$\dfrac{2a^3}{(s^2+a^2)^2}$
22. $t\sinh at$	$\dfrac{2as}{(s^2-a^2)^2}$
23. $t\cosh at$	$\dfrac{s^2+a^2}{(s^2-a^2)^2}$
24. $at\cosh at - \sinh at$	$\dfrac{2a^3}{(s^2-a^2)^2}$
25. $e^{-bt}\sin at$	$\dfrac{a}{(s+b)^2+a^2}$
26. $e^{-bt}\cos at$	$\dfrac{s+b}{(s+b)^2+a^2}$
27. $\sin at\cosh at - \cos at\sinh at$	$\dfrac{4a^3}{(s^4+4a^4)}$
28. $\sin at\sinh at$	$\dfrac{2a^2s}{(s^4+4a^4)}$
29. $\sin at\cosh at - \cos at\sinh at$	$\dfrac{2as^2}{s^4+4a^4}$
30. $\cos at\cosh at$	$\dfrac{s^3}{s^4+4a^4}$
31. $\sinh at - \sin at$	$\dfrac{2a^3}{s^4-a^4}$
32. $\cosh at - \cos at$	$\dfrac{2a^2s}{s^4-a^4}$

Appendix G: Contour Integration and Inverse Laplace Transform

G.1 Analytic Functions of a Complex Variable

A function f of the complex variable z is analytic at a point z_0 if its derivative $f'(z_0)$ exists not only at z_0 but at every point z in some neighborhood of z_0. It is analytic in a domain of the z plane if it is analytic at every point in that domain.

An entire function is one that is analytic at every point of the z plane throughout the entire plane. For example, every polynomial is an entire function:

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad n = 0, 1, 2, \ldots$$

If a function is analytic at some point in every neighborhood of a point z_0 except at z_0 itself, then z_0 is called a singular point, or a singularity, of the function. For example,

$$f(z) = \frac{1}{z}$$

then

$$f'(z) = -\frac{1}{z^2} \quad (z \neq 0)$$

The point $z = 0$ is a singular point.

Properties of Analytic Functions

1) If two functions are analytic in a domain D, their sum and their product are both analytic in D. Their quotient is analytic in D provided that the function in the demoninator does not vanish at any point in D.

2) An analytic function of another analytic function is analytic.

Conditions of Analytic Functions (Cauchy–Riemann Equations)

Suppose that a function $f(z)$ can be written as

$$f(z) = u(x, y) + iv(x, y) \tag{G.1}$$

The conditions for the function to be analytic at point z_0 are that the following two equations must be satisfied at that point:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{G.2}$$

These are known as Cauchy–Riemann equations.

G.2 Line Integrals of Complex Functions

If C is a curve in the complex plane joining the points z_0 and z_1, the line integral of a function $f(z) = u(x, y) + iv(x, y)$ along C is defined by the equation

$$\int_C f(z)\mathrm{d}z = \int_C (u + iv)(\mathrm{d}x + i\mathrm{d}y)$$

$$= \int_C [(u\mathrm{d}x - v\mathrm{d}y) + i(v\mathrm{d}x + u\mathrm{d}y)] \tag{G.3}$$

Without proof, we state some facts as follows:

1) The line integral of an analytic function is independent of path.

2) If a function $f(z)$ is analytic at all points interior to and on a closed contour C, then

$$\int_C f(z)\mathrm{d}z = 0 \tag{G.4}$$

Now let us consider the following integration

$$\oint_{C_1} f(z)\mathrm{d}z = \oint_{C_1} \frac{1}{z}\mathrm{d}z$$

where C_1 is the unit circle $|z| = 1$ with the center at the origin. Notice that $f(z) = 1/z$ is analytic everywhere in the complex plane except the origin. Evaluating this line integral, we find

$$\oint_{C_1} \frac{1}{z}\mathrm{d}z = \int_0^{2\pi} e^{-i\theta}(ie^{i\theta}\mathrm{d}\theta) = \int_0^{2\pi} i\mathrm{d}\theta = 2\pi i \tag{G.5}$$

More generally, let us consider any other closed curve C that surrounds the origin. If we make a "crosscut" from C to C_1 and in the region R, $f(z) = 1/z$ is analytic everywhere, and we have

$$\oint_C \frac{\mathrm{d}z}{z} + \oint_{C_1} \frac{\mathrm{d}z}{z} = 0$$

$$\oint_C \frac{\mathrm{d}z}{z} = -\oint_{C_1} \frac{\mathrm{d}z}{z} = -\int_{2\pi}^{0} i\mathrm{d}\theta = 2\pi i \tag{G.6}$$

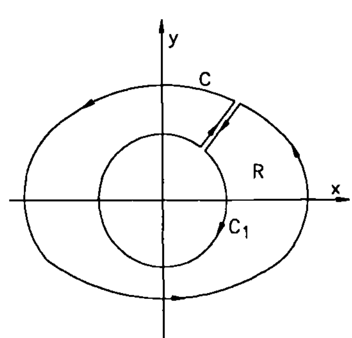


Fig. G.1 Contour integral around a singular point at $z = 0$.

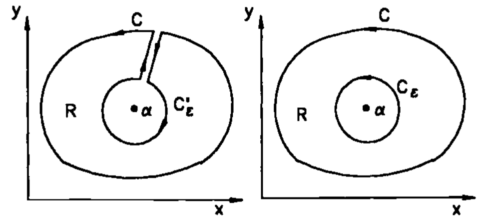


Fig. G.2 Contour integral around a singular point at $z = \alpha$.

Therefore, any closed curve surrounding the origin will have the result of $2\pi i$.

Cauchy's Integral Formula

Let C be a closed contour, inside which and along C, $f(z)$ is analytic, and let α be a point inside C. Furthermore let C_ϵ be a small circle of radius ϵ, with center at the point α as shown in Fig. G.2. Then, clearly the function $f(z)/(z-\alpha)$ is analytic in R so that

$$\oint_C \frac{f(z)}{z-\alpha}\mathrm{d}z + \int_{c'_\epsilon} \frac{f(z)}{z-\alpha}\mathrm{d}z = 0$$

$$\oint_C \frac{f(z)}{z-\alpha}\mathrm{d}z = -\int_{c'_\epsilon} \frac{f(z)}{z-\alpha}\mathrm{d}z = \int_{c_\epsilon} \frac{f(z)}{z-\alpha}\mathrm{d}z$$

Note that the directions of line integrals on c_ϵ and c'_ϵ are different. On the circle c_ϵ

$$z = \alpha + \epsilon e^{i\theta} \qquad \mathrm{d}z = i\epsilon e^{i\theta}\mathrm{d}\theta \qquad \oint_{c_\epsilon} \frac{f(z)}{z-\alpha}\mathrm{d}z = i\int_0^{2\pi} f(\alpha + \epsilon e^{i\theta})\mathrm{d}\theta$$

In the limit $\epsilon \to 0$,

$$\oint_C \frac{f(z)}{z-\alpha}\mathrm{d}z = \lim_{\epsilon\to 0} i\int_0^{2\pi} f(\alpha + \epsilon e^{i\theta})\mathrm{d}\theta = 2\pi i f(\alpha)$$

Therefore

$$f(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}\mathrm{d}z \tag{G.7}$$

This is known as Cauchy's integral formula.

Example G.1

Evaluate $\oint_C \frac{z^2+1}{z^2-1}\mathrm{d}z$ if C is a circle of unit radius with the center at $z = 1$ and then at $z = -1$.

Solution. Consider the integral as

$$I_1 = \oint_{C_1} \frac{z^2+1}{z+1}\,\frac{\mathrm{d}z}{z-1}$$

i.e.,

$$f_1(z) = \frac{z^2+1}{z+1} \quad \text{and} \quad \alpha = 1$$

Hence

$$I_1 = 2\pi i f_1(1) = 2\pi i$$

Consider also

$$I_2 = \oint_{C_2} \frac{z^2+1}{z-1}\frac{\mathrm{d}z}{z+1}$$

$$f_2(z) = \frac{z^2+1}{z-1} \qquad \text{and} \qquad \alpha = -1$$

Hence

$$I_2 = 2\pi i f_2(-1) = -2\pi i$$

Poles and Residues

If $z = a$ is an isolated singular point of $f(z)$, but if for some integer m the product

$$(z-a)^m f(z)$$

is analytic at $z = a$, then $f(z)$ has a pole at $z = a$. If m is the smallest integer for which this is so, the pole is said to be of order m.

Now suppose that the analytic function $f(z)$ has a pole of order m at the point $z = a$. Then $(z-a)^m f(z)$ is analytic and hence can be expanded into the Taylor series as

$$(z-a)^m f(z) = A_0 + A_1(z-a) + \cdots + A_{m-1}(z-a)^{m-1} + A_m(z-a)^m + \cdots \tag{G.8}$$

or

$$f(z) = \frac{A_0}{(z-a)^m} + \frac{A_1}{(z-a)^{m-1}} + \cdots + \frac{A_{m-1}}{z-a} + A_m + A_{m+1}(z-a) + \cdots \tag{G.9}$$

Let C_a be any closed contour surrounding $z = a$ that lies inside the circle of convergence of Eq. (G.8) and which is such that $f(z)$ is analytic inside and on C_a, except at $z = a$. If we integrate Eq. (G.9) around this contour, we have

$$\oint_{C_a} f(z)\mathrm{d}z = \oint_{C_a}\left[\frac{A_0}{(z-a)^m} + \frac{A_1}{(z-a)^{m-1}} + \cdots + \frac{A_{m-1}}{(z-a)} + \cdots\right]\mathrm{d}z \tag{G.10}$$

Note that

$$\oint_{C_a} (z-a)^n \mathrm{d}z = \epsilon \int_0^{2\pi} e^{ni\theta}(ie^{i\theta}\mathrm{d}\theta) = \epsilon i \int_0^{2\pi} e^{i(n+1)\theta}\mathrm{d}\theta$$

$$= \epsilon i \int_0^{2\pi} [\cos(n+1)\theta + i\sin(n+1)\theta]\mathrm{d}\theta = 0 \quad \text{if } n \neq -1 \qquad \text{(G.11)}$$

and

$$\oint_{C_a} \frac{1}{z-a}\mathrm{d}z = 2\pi i \qquad \text{(G.12)}$$

Using the results given, we find

$$\oint_{C_a} f(z)\mathrm{d}z = 2\pi i A_{m-1} \qquad \text{(G.13)}$$

We call the coefficient A_{M-1} the residue of $f(z)$ at $z = a$ and denote it by Res(a). Hence, with the use of the expression for the coefficients of the Taylor series, we have

$$\text{Res}(a) = A_{m-1} = \frac{1}{(m-1)!}\left\{\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}[(z-a)^m f(z)]\right\}_{z=a} \qquad \text{(G.14)}$$

In the case of a simple pole ($m = 1$), from Eq. (G.14), we find

$$\text{Res(a)} = [(z-a)f(z)]_{z=a} = \lim_{z\to a}[(z-a)f(z)] \qquad \text{(G.15)}$$

In the case, if $f(z)$ is expressed as the ratio of two functions

$$f(z) = \frac{N(z)}{D(z)}$$

but it has a simple pole at $z = a$, then

$$\text{Res(a)} = \lim_{z\to a}\left[\frac{(z-a)}{D(z)}N(z)\right]$$

$$= \frac{N(a)}{D'(a)} \qquad \text{(G.16)}$$

Suppose now that C is the boundary of a finite region inside which $f(z)$ is single-valued and has only isolated singularities at a finite number of point $z = a_1, a_2, \ldots, a_n$. We enclose these points by small circles $c_1, c_2, \ldots, c_n$. Then

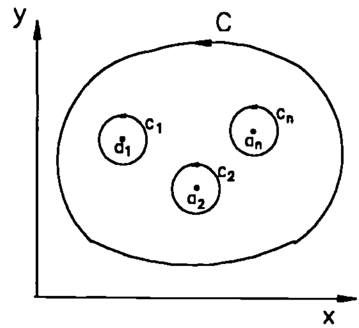


Fig. G.3 Finite number of singularities enclosed in *C*.

by introducing a crosscut from each circle to C, a simply connected region R is obtained inside which $f(z)$ is analytic. Thus we have

$$\oint_C f(z)\mathrm{d}z = \sum_{j=1}^{n} \oint_{c_j} f(z)\mathrm{d}z = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(a_j) \qquad \text{(G.17)}$$

This result is known as Cauchy's residue theorem.

G.3 Contour Integrals

Before we apply the method of contour integrals, we must make statements about the following theorems.

Theorem I

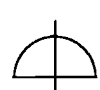
If, on a circular arc C_R with radius$R$ and center at the origin, $zf(z) \to 0$ uniformly as $R \to \infty$, then

$$\lim_{R\to\infty} \int_{C_R} f(z)\mathrm{d}z = 0$$

Theorem II

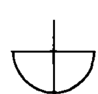
Suppose that, on a circular arc C_R with radius R and center at origin, $f(z) \to 0$ uniformly as $R \to \infty$. Then

1) $$\lim_{R\to\infty} \int_{C_R} e^{iuz} f(z)\mathrm{d}z = 0 \qquad \text{as } u > 0$$



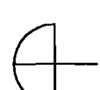
where C_R is in the first and/or second quadrants.

2) $$\lim_{R\to\infty} \int_{C_R} e^{iuz} f(z)\mathrm{d}z = 0 \qquad \text{as } u < 0$$



where C_R is in the third and/or fourth quadrants.

3) $$\lim_{R\to\infty} \int_{C_R} e^{sz} f(z)\mathrm{d}z = 0 \qquad \text{as } s > 0$$



where C_R is in the second and/or third quadrants.

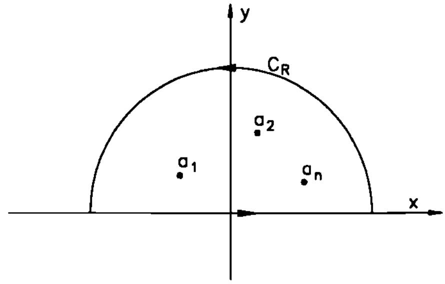
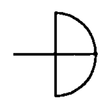


Fig. G.4 Finite number of singularities enclosed in the contour.

4) $\lim_{R\to\infty} \int_{C_R} e^{sz} f(z)\mathrm{d}z = 0 \qquad s \le 0$



where C_R is in the first and/or fourth quadrants.

Now let us apply the residue theorem to evaluate some improper real integrals such as

$$I = \int_{-\infty}^{\infty} \frac{P(x)}{q(x)}\mathrm{d}x \tag{G.18}$$

where p and q are polynomials with no factors in common. We can replace the variable x with the complex variable z and choose the contour as shown in Fig. G.4. Hence we have

$$\int_{-\infty}^{\infty} \frac{p(z)}{q(z)}\mathrm{d}z + \int_{C_R} \frac{p(z)}{q(z)}\mathrm{d}z = 2\pi i \sum_j \mathrm{Res}(a_j)$$

Note that the first term in the preceding equation means integrating along $y = 0$; the second part is proved to be vanishing on C_R as given in the example. Then we have

$$I = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(a_j)$$

Example G.2

Evaluate

$$I = \int_{0}^{\infty} \frac{\mathrm{d}x}{x^2+1}$$

Solution. Because the integrand is an even function of x, we have

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+1}$$

Choose the contour as shown in Fig. G.4. The contour integral can be written as

$$\lim_{R\to\infty}\left[\int_{-R}^{R}\frac{\mathrm{d}x}{x^2+1}+\int_{C_R}\frac{\mathrm{d}z}{z^2+1}\right]=2\pi i\sum_j \mathrm{Res}(a_j)$$

Because

$$f(z)=\frac{1}{z^2+1}=\frac{1}{(z+i)(z-i)}$$

there is only one simple pole $z=i$ within the contour.

$$\mathrm{Res}(i)=\left.\frac{1}{z+i}\right|_{z=i}=\frac{1}{2i}$$

On C_R, we have

$$|z^2+1|>|z^2|-1=R^2-1$$

$$\lim_{R\to 0}\left|\int_{C_R}f(z)\mathrm{d}z\right|=\lim_{R\to\infty}\left|\int_{C_R}\frac{\mathrm{d}z}{z^2+1}\right|\le\lim_{R\to\infty}\left|\int_{C_R}\left|\frac{1}{z^2+1}\right||\mathrm{d}z|\right|$$

$$<\lim_{R\to\infty}\int_{C_R}\frac{|\mathrm{d}z|}{R^2-1}=\lim_{R\to\infty}\frac{\pi R}{R^2-1}=0$$

Therefore

$$I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2+1}=\frac{1}{2}\left[2\pi i\frac{1}{2i}\right]=\frac{\pi}{2}$$

Example G.3

Evaluate

$$I=\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\frac{e^{iu\lambda}}{u-i\alpha}\mathrm{d}u \qquad \text{as } \lambda\ge 0$$

Also find the value of I as $\lambda<0$.

Solution. Choose the contour as shown in Fig. G.5 for the case $\lambda>0$.

$$I=\lim_{\alpha\to 0}2\pi i(e^{iu\lambda})_{u=i\alpha}=2\pi i$$

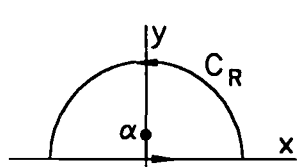


Fig. G.5 One singularity enclosed in the contour.

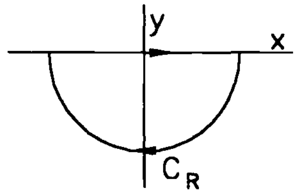


Fig. G.6 No pole enclosed in the contour.

Because of Theorem II.1, the integral over $C_R = 0$.

For $\lambda < 0$, because of Theorem II.2, we choose the contour as shown in Fig. G.6. There is no pole in this contour. Therefore

$$I = 0 \qquad \text{as } \lambda < 0$$

G.4 Inverse Laplace Transform

The method of the contour integral also can be used to evaluate the inverse Laplace transformation. Recall Eq. (8.37) for the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\mathrm{d}s$$

We illustrate the procedure for the method through the following example.

Example G.4

Find $f(t)$ from $F(s) = s/(s^2 + k^2)$ through the evaluation of the complex inversion integral

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-\mathrm{i}\infty}^{\gamma+i\infty} \frac{s}{s^2+k^2}e^{st}\,\mathrm{d}s$$

Solution. Choose the contour as shown in Fig. G.7. The contour integral can be written as

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{s}{s^2+k^2}e^{st}\mathrm{d}s + \frac{1}{2\pi i}\int_{C_2} \frac{s}{s^2+k^2}e^{st}\mathrm{d}s = \sum_j \quad \text{Res} \quad \text{at } s_j = \pm ik$$

Evaluating the residues, we have the following: At $s = ik$,

$$\lim_{s\to ik}\left[(s-ik)\frac{se^{st}}{(s-ik)(s+ik)}\right] = \frac{ike^{ikt}}{2ik} = \frac{1}{2}e^{ikt}$$

At $s = -ik$,

$$\lim_{s\to -ik}\left[(s+ik)\frac{se^{st}}{(s-ik)(s+ik)}\right] = \frac{-ike^{-ikt}}{-2ik} = \frac{1}{2}e^{-ikt}$$

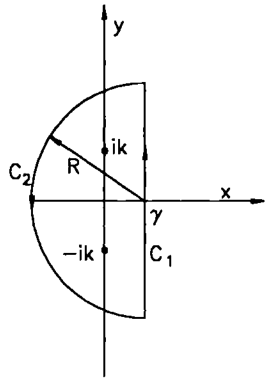


Fig. G.7 Contour for inverse Laplace transform.

It can be proved that the line integral over C_2 is zero. Therefore

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{s}{s^2+k^2} e^{st} \mathrm{d}s = \frac{1}{2}(e^{ikt} + e^{-ikt}) = \cos kt$$

Appendix H: Bessel Functions

H.1 Bessel Equation and Its Series Solutions

The Bessel equation of order n with a parameter λ can be written as

$$r^2R'' + rR' + (\lambda^2r^2 - n^2)R = 0 \tag{H.1}$$

where λ is a real number. With the change of variable

$$x = \lambda r$$

Eq. (H.1) becomes

$$x^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + x\frac{\mathrm{d}R}{\mathrm{d}x} + (x^2 - n^2)R = 0 \tag{H.2}$$

If n is not an integer, the solution of Eq. (H.2) is

$$R(x) = c_1J_n(x) + c_2J_{-n}(x) \tag{H.3}$$

where

$$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{n+2m}}{2^{2m+n}m!\Gamma(n+m+1)} \tag{H.4}$$

$J_n(x)$ is known as the Bessel function of the first kind of order n. Notice that the function is an infinite series. In the denominator there is a gamma function denoted by $\Gamma(n+m+1)$. Similarly we have

$$J_{-n}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{-n+2m}}{2^{2m-n}m!\Gamma(-n+m+1)} \tag{H.5}$$

When n is an integer, it can be proved that

$$J_{-n}(x) = (-1)^nJ_n(x)$$

Hence, $J_n(x)$ and $J_{-n}(x)$ are not independent. The general solution of Eq. (H.2) becomes

$$R(x) = c_1J_n(x) + c_2Y_n(x) \tag{H.6}$$

$Y_n(x)$ is the Bessel function of the second kind of order n. Because $Y_n(x)$ becomes infinite as $x \to 0$, c_2, is usually set to zero. The detail expression for $Y_n(x)$ is

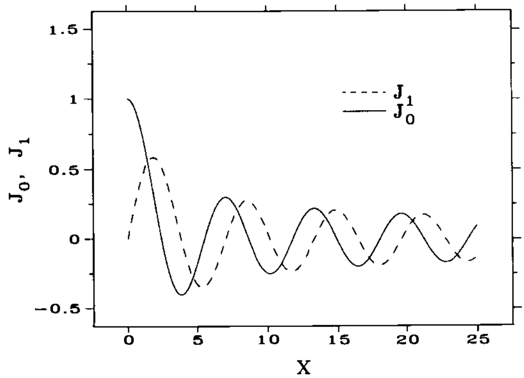


Fig. H.1 Bessel functions J_0, J_1.

omitted here. Interested readers can refer to the "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables" by the National Bureau of Standards. The gamma function in Eq. (H.4) may be briefly described as

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, \mathrm{d}t \tag{H.7}$$

When x = integer $= n$,

$$\Gamma(n+1) = n! \tag{H.8}$$

The graphs of Bessel functions are shown in Figs. (H.1) and (H.2). Numerical values of Bessel functions of order of 0 and 1 are given in the tables, at the end of this appendix.

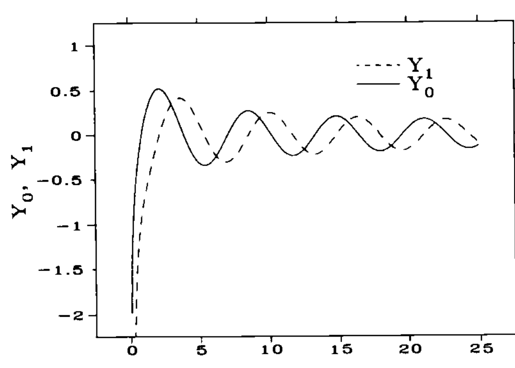


Fig. H.2 Bessel functions Y_0, Y_1.

H.2 Properties of Bessel Functions

The following formulas are collected here for the readers' convenience. Detailed proofs are omitted.

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{n+1}J_{x+1}(x)] = x^{n+1}J_n(x) \tag{H.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x) \tag{H.10}$$

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x) \tag{H.11}$$

$$J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x) \tag{H.12}$$

$$\int x^{n+1}J_n(x)\,\mathrm{d}x = x^{n+1}J_{n+1}(x) + c \tag{H.13}$$

$$\int x^{-n}J_{n+1}(x)\,\mathrm{d}x = -x^{-n}J_n(x) + c \tag{H.14}$$

$$\cos(x\sin\phi) = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos 2k\phi \tag{H.15}$$

$$\sin(x\sin\phi) = 2\sum_{k=1}^{\infty} J_{2k-1}(x)\sin(2k-1)\phi \tag{H.16}$$

$$\int_0^{\pi} \cos n\phi \cos(x\sin\phi)\,\mathrm{d}\phi = \begin{cases} \pi J_n(x) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{H.17}$$

$$\int_0^{\pi} \sin n\phi \sin(x\sin\phi)\,\mathrm{d}\phi = \begin{cases} 0 & n \text{ even} \\ \pi J_n(x) & n \text{ odd} \end{cases} \tag{H.18}$$

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} [\cos n\phi\cos(x\sin\phi) + \sin n\phi\sin(x\sin\phi)]\,\mathrm{d}\phi \tag{H.19}$$

In the preceding expressions, n is an integer.

H.3 Fourier–Bessel Series

A function can be expanded into a Fourier–Bessel series as

$$f(r) = A_1J_k(\lambda_1 r) + A_2J_k(\lambda_2 r) + \cdots + A_nJ_k(\lambda_n r) + \cdots + \qquad \text{as } 0 \le r \le b \tag{H.20}$$

The conditions for the function to be expressed by the preceding equation may be stated without proof as follows. The function $f(r)$ must be piecewise continuous. The Bessel function $J_k(\lambda r)$ satisfies the condition

$$A J_k(\lambda b) - B \left. \frac{\mathrm{d}J_k(\lambda r)}{\mathrm{d}r} \right|_b = 0 \tag{H.21}$$

where A, B are constants. Then

$$A_n = \frac{\int_0^b f(r) r J_k(\lambda_{nr}) \mathrm{d}r}{\int_0^b r J_k^2(\lambda_n r) \mathrm{d}r} \tag{H.22}$$

where

$$\int_0^b r J_k^2(\lambda_n r) \mathrm{d}r = \begin{cases} \frac{J_k^2(\lambda_n b)}{2\lambda_n^2}\left[(\lambda_n b)^2 - k^2 + \left(\frac{bA}{B}\right)^2\right] & B \neq 0 \\ \frac{b^2}{2} J_{k+1}^2(\lambda_n b) & B = 0 \end{cases} \tag{H.23}$$

Example H.1

Expand $f(r) = r^2$ over the interval $0 < r < 3$ in terms of the function $J_0(\lambda_n r)$ where the λ_n are determined by $J_1(3\lambda) = 0$.

Solution. The roots of $J_1(3\lambda) = 0$ are

$$3\lambda_0 = 0, \qquad 3\lambda_1 = 3.832, \qquad 3\lambda_2 = 7.016$$

$$3\lambda_3 = 10.174, \qquad 3\lambda_4 = 13.324, \ldots$$

or

$$\lambda_0 = 0, \qquad \lambda_1 = 1.277, \qquad \lambda_2 = 2.339, \qquad \lambda_3 = 3.391, \qquad \lambda_4 = 4.441$$

Looking into the boundary condition Eq. (H.21), we have

$$A J_0(3\lambda) - B \frac{\mathrm{d}J_0(\lambda r)}{\mathrm{d}r} = 0$$

because

$$\frac{\mathrm{d}J_0(x)}{\mathrm{d}x} = -J_1(x) = 0$$

and $J_0(3\lambda) \neq 0$. The constant A must be zero because of the boundary condition. The coefficients

$$A_n = \frac{\int_0^3 r^3 J_0(\lambda_n r) \mathrm{d}r}{\int_0^3 r J_0^2(\lambda_n r) \mathrm{d}r}$$

Using Eq. (H.23), we find

$$\int_0^3 r J_0^2(\lambda_n r)\,\mathrm{d}r = \frac{3^2}{2} J_0^2(\lambda_n b)$$

To integrate the numerator in A_n,

$$I = \int_0^3 r^3 J_0(\lambda_n r)\,\mathrm{d}r = \frac{1}{\lambda_n^4} \int_0^{3\lambda_n} z^3 J_0(z)\,\mathrm{d}z$$

We use integration by parts, Eq. (H.13), and find

$$I = -\frac{18}{\lambda_n^2} J_2(3\lambda_n)$$

Hence

$$A_n = \frac{\left(-18/\lambda_n^2\right) J_2(3\lambda_n)}{(9/2) J_0^2(3\lambda_n)} = -\frac{4}{\lambda_n^2} \frac{J_2(3\lambda_n)}{J_0^2(3\lambda_n)} = \frac{4}{\lambda_n^2 J_0(3\lambda_n)}$$

For $n = 0$, $\lambda_0 = 0$, and $J_0(0) = 1$,

$$A_0 = \frac{\int_0^3 r^3\,\mathrm{d}r}{\int_0^3 r\,\mathrm{d}r} = \frac{9}{2}$$

Therefore, the Fourier–Bessel series is

$$r^2 = \frac{9}{2} + \sum_{n=1}^{\infty} \frac{4}{\lambda_n^2 J_0(3\lambda_n)} J_0(\lambda_n r) \qquad \text{as } 0 \le r \le 3$$

Table H.1 Bessel functions of integer order

X	J_0	J_1	Y_0	Y_1
0.00	0.100000E+01	0.000000E+00	$-\infty$	$-\infty$
0.05	0.999375E+00	0.249922E−01	−0.197931E+01	−0.127899E+02
0.10	0.997502E+00	0.499375E−01	−0.153424E+01	−0.645895E+01
0.15	0.994383E+00	0.747893E−01	−0.127078E+01	−0.436368E+01
0.20	0.990025E+00	0.995008E−01	−0.108111E+01	−0.332382E+01
0.25	0.984436E+00	0.124026E+00	−0.931573E+00	−0.270411E+01
0.30	0.977626E+00	0.148319E+00	−0.807274E+00	−0.229311E+01
0.35	0.969609E+00	0.172334E+00	−0.700314E+00	−0.200040E+01
0.40	0.960398E+00	0.196027E+00	−0.606025E+00	−0.178087E+01
0.45	0.950012E+00	0.219353E+00	−0.521428E+00	−0.160954E+01
0.50	0.938470E+00	0.242268E+00	−0.444519E+00	−0.147147E+01
0.55	0.925793E+00	0.264732E+00	−0.373886E+00	−0.135718E+01
0.60	0.912005E+00	0.286701E+00	−0.308510E+00	−0.126039E+01
0.65	0.897132E+00	0.308135E+00	−0.247628E+00	−0.117677E+01
0.70	0.881201E+00	0.328996E+00	−0.190665E+00	−0.110325E+01
0.75	0.864242E+00	0.349244E+00	−0.137173E+00	−0.103759E+01
0.80	0.846287E+00	0.368842E+00	−0.868021E−01	−0.978144E+00
0.85	0.827369E+00	0.387755E+00	−0.392759E−01	−0.923643E+00
0.90	0.807524E+00	0.405950E+00	0.562846E−02	−0.873126E+00
0.95	0.786787E+00	0.423392E+00	0.480906E−01	−0.825846E+00
1.00	0.765198E+00	0.440051E+00	0.882571E−01	−0.781213E+00
1.05	0.742796E+00	0.455897E+00	0.126248E+00	−0.738761E+00
1.10	0.719622E+00	0.470902E+00	0.162163E+00	−0.698120E+00
1.15	0.695720E+00	0.485041E+00	0.196085E+00	−0.658991E+00
1.20	0.671133E+00	0.498289E+00	0.228083E+00	−0.621136E+00

1.25	0.645906E+00	0.510623E+00	0.258217E+00	−0.584364E+00
1.30	0.620086E+00	0.522023E+00	0.286535E+00	−0.548520E+00
1.35	0.593720E+00	0.532470E+00	0.313082E+00	−0.513480E+00
1.40	0.566855E+00	0.541948E+00	0.337895E+00	−0.479147E+00
1.45	0.539541E+00	0.550441E+00	0.361007E+00	−0.445443E+00
1.50	0.511828E+00	0.557936E+00	0.382449E+00	−0.412309E+00
1.55	0.483765E+00	0.564424E+00	0.402247E+00	−0.379698E+00
1.60	0.455402E+00	0.569896E+00	0.420427E+00	−0.347578E+00
1.65	0.426792E+00	0.574344E+00	0.437012E+00	−0.315926E+00
1.70	0.397985E+00	0.577765E+00	0.452027E+00	−0.284727E+00
1.75	0.369033E+00	0.580156E+00	0.465492E+00	−0.253973E+00
1.80	0.339987E+00	0.581517E+00	0.477432E+00	−0.223665E+00
1.85	0.310898E+00	0.581849E+00	0.487866E+00	−0.193807E+00
1.90	0.281819E+00	0.581157E+00	0.496820E+00	−0.164406E+00
1.95	0.252800E+00	0.579446E+00	0.504315E+00	−0.135476E+00
2.00	0.223891E+00	0.576725E+00	0.510376E+00	−0.107033E+00
2.05	0.195144E+00	0.573003E+00	0.515027E+00	−0.790937E−01
2.10	0.166607E+00	0.568292E+00	0.518294E+00	−0.516791E−01
2.15	0.138331E+00	0.562607E+00	0.520204E+00	−0.248110E−01
2.20	0.110363E+00	0.555963E+00	0.520784E+00	0.148734E−02
2.25	0.827504E−01	0.548379E+00	0.520065E+00	0.271916E−01
2.30	0.555403E−01	0.539873E+00	0.518075E+00	0.522768E−01
2.35	0.287781E−01	0.530467E+00	0.514848E+00	0.767177E−01
2.40	0.250828E−02	0.520186E+00	0.510415E+00	0.100488E+00
2.45	−0.232262E−01	0.509052E+00	0.504811E+00	0.123564E+00
2.50	−0.483832E−01	0.497094E+00	0.498071E+00	0.145918E+00
2.55	−0.729223E−01	0.484340E+00	0.490231E+00	0.167526E+00

(Cont.)

Table H.1 Bessel functions of integer order (continued)

X	J_0	J_1	Y_0	Y_1
2.60	−0.968043E−01	0.470819E+00	0.481331E+00	0.188363E+00
2.65	−0.119992E+00	0.456562E+00	0.471408E+00	0.208406E+00
2.70	−0.142449E+00	0.441602E+00	0.460504E+00	0.227632E+00
2.75	−0.164141E+00	0.425973E+00	0.448659E+00	0.246018E+00
2.80	−0.185035E+00	0.409710E+00	0.435916E+00	0.263545E+00
2.85	−0.205102E+00	0.392849E+00	0.422319E+00	0.280191E+00
2.90	−0.224311E+00	0.375428E+00	0.407912E+00	0.295940E+00
2.95	−0.242636E+00	0.357485E+00	0.392741E+00	0.310772E+00
3.00	−0.260051E+00	0.339060E+00	0.376851E+00	0.324674E+00
3.05	−0.276534E+00	0.320192E+00	0.360289E+00	0.337630E+00
3.10	−0.292064E+00	0.300922E+00	0.343103E+00	0.349629E+00
3.15	−0.306621E+00	0.281292E+00	0.325342E+00	0.360659E+00
3.20	−0.320188E+00	0.261344E+00	0.307054E+00	0.370711E+00
3.25	−0.332750E+00	0.241120E+00	0.288288E+00	0.379777E+00
3.30	−0.344296E+00	0.220664E+00	0.269093E+00	0.387853E+00
3.35	−0.354814E+00	0.200019E+00	0.249519E+00	0.394933E+00
3.40	−0.364295E+00	0.179227E+00	0.229616E+00	0.401015E+00
3.45	−0.372735E+00	0.158332E+00	0.209434E+00	0.406100E+00
3.50	−0.380127E+00	0.137378E+00	0.189023E+00	0.410188E+00
3.55	−0.386472E+00	0.116409E+00	0.168432E+00	0.413284E+00
3.60	−0.391769E+00	0.954665E−01	0.147711E+00	0.415392E+00
3.65	−0.396020E+00	0.745944E−01	0.126909E+00	0.416519E+00
3.70	−0.399230E+00	0.538350E−01	0.106075E+00	0.416674E+00
3.75	−0.401406E+00	0.332303E−01	0.852578E−01	0.415869E+00
3.80	−0.402556E+00	0.128220E−01	0.645042E−01	0.414115E+00

3.85	−0.402692E+00	−0.734920E−02	0.438619E−01	0.411427E+00
3.90	−0.401826E+00	−0.272430E−01	0.233770E−01	0.407820E+00
3.95	−0.399973E+00	−0.468203E−01	0.309490E−02	0.403314E+00
4.00	−0.397150E+00	−0.660423E−01	−0.169397E−01	0.397926E+00
4.05	−0.393375E+00	−0.848719E−01	−0.366833E−01	0.391679E+00
4.10	−0.388670E+00	−0.103272E+00	−0.560937E−01	0.384594E+00
4.15	−0.383056E+00	−0.121209E+00	−0.751293E−01	0.376697E+00
4.20	−0.376557E+00	−0.138646E+00	−0.937504E−01	0.368013E+00
4.25	−0.369200E+00	−0.155553E+00	−0.111918E+00	0.358569E+00
4.30	−0.361011E+00	−0.171896E+00	−0.129595E+00	0.348394E+00
4.35	−0.352020E+00	−0.187646E+00	−0.146746E+00	0.337518E+00
4.40	−0.342257E+00	−0.202775E+00	−0.163336E+00	0.325971E+00
4.45	−0.331753E+00	−0.217255E+00	−0.179333E+00	0.313786E+00
4.50	−0.320543E+00	−0.231060E+00	−0.194705E+00	0.300998E+00
4.55	−0.308659E+00	−0.244167E+00	−0.209423E+00	0.287639E+00
4.60	−0.296138E+00	−0.256553E+00	−0.223460E+00	0.273745E+00
4.65	−0.283016E+00	−0.268197E+00	−0.236789E+00	0.259354E+00
4.70	−0.269331E+00	−0.279081E+00	−0.249388E+00	0.244501E+00
4.75	−0.255121E+00	−0.289187E+00	−0.261232E+00	0.229226E+00
4.80	−0.240425E+00	−0.298500E+01	−0.272304E+00	0.213565E+00
4.85	−0.225284E+00	−0.307006E+00	−0.282583E+00	0.197559E+00
4.90	−0.209738E+00	−0.314695E+00	−0.292055E+00	0.181247E+00
4.95	−0.193828E+00	−0.321555E+00	−0.300704E+00	0.164668E+00
5.00	−0.177596E+00	−0.327579E+00	−0.308518E+00	0.147863E+00
5.05	−0.161084E+00	−0.332761E+00	−0.315487E+00	0.130872E+00
5.10	−0.144334E+00	−0.337097E+00	−0.321603E+00	0.113736E+00
5.15	−0.127389E+00	−0.340585E+00	−0.326859E+00	0.964951E−01

(Cont.)

Table H.1 Bessel functions of integer order (continued)

X	J_0	J_1	Y_0	Y_1
5.20	−0.110290E+00	−0.343223E+00	−0.331251E+00	0.791897E−01
5.25	−0.930803E−01	−0.345014E+00	−0.334777E+00	0.618603E−01
5.30	−0.758024E−01	−0.345961E+00	−0.337437E+00	0.445469E−01
5.35	−0.584981E−01	−0.346069E+00	−0.339233E+00	0.272892E−01
5.40	−0.412093E−01	−0.345345E+00	−0.340168E+00	0.101264E−01
5.45	−0.239772E−01	−0.343798E+00	−0.340248E+00	−0.690248E−02
5.50	−0.684286E−02	−0.341438E+00	−0.339481E+00	−0.237592E−01
5.55	0.101534E−01	−0.338278E+00	−0.337875E+00	−0.404062E−01
5.60	0.269720E−01	−0.334333E+00	−0.335444E+00	−0.568067E−01
5.65	0.435739E−01	−0.329616E+00	−0.332199E+00	−0.729245E−01
5.70	0.599211E−01	−0.324147E+00	−0.328157E+00	−0.887244E−01
5.75	0.759766E−01	−0.317944E+00	−0.323333E+00	−0.104172E+00
5.80	0.917038E−01	−0.311027E+00	−0.317746E+00	−0.119235E+00
5.85	0.107068E+00	−0.303419E+00	−0.311416E+00	−0.133881E+00
5.90	0.122035E+00	−0.295142E+00	−0.304365E+00	−0.148078E+00
5.95	0.136571E+00	−0.286221E+00	−0.296616E+00	−0.161798E+00
6.00	0.150647E+00	−0.276683E+00	−0.288194E+00	−0.175012E+00
6.05	0.164230E+00	−0.266554E+00	−0.279124E+00	−0.187692E+00
6.10	0.177293E+00	−0.255864E+00	−0.269434E+00	−0.199813E+00
6.15	0.189808E+00	−0.244640E+00	−0.259152E+00	−0.211352E+00
6.20	0.201748E+00	−0.232915E+00	−0.248309E+00	−0.222285E+00
6.25	0.213091E+00	−0.220719E+00	−0.236934E+00	−0.232591E+00
6.30	0.223813E+00	−0.208085E+00	−0.225060E+00	−0.242251E+00
6.35	0.233893E+00	−0.195046E+00	−0.212720E+00	−0.251246E+00
6.40	0.243312E+00	−0.181636E+00	−0.199947E+00	−0.259561E+00

6.45	0.252051*E*+00	−0.167888*E*+00	−0.186775*E*+00	−0.267181*E*+00
6.50	0.260096*E*+00	−0.153839*E*+00	−0.173241*E*+00	−0.274092*E*+00
6.55	0.267431*E*+00	−0.139524*E*+00	−0.159378*E*+00	−0.280284*E*+00
6.60	0.274044*E*+00	−0.124978*E*+00	−0.145224*E*+00	−0.285748*E*+00
6.65	0.279925*E*+00	−0.110238*E*+00	−0.130816*E*+00	−0.290475*E*+00
6.70	0.285065*E*+00	−0.953399*E*−01	−0.116189*E*+00	−0.294460*E*+00
6.75	0.289457*E*+00	−0.083205*E*−01	−0.101382*E*+00	−0.297698*E*+00
6.80	0.293096*E*+00	−0.652163*E*−01	−0.864316*E*−01	−0.300187*E*+00
6.85	0.295978*E*+00	−0.500638*E*−01	−0.713755*E*−01	−0.301927*E*+00
6.90	0.298102*E*+00	−0.348996*E*−01	−0.562512*E*−01	−0.302918*E*+00
6.95	0.299469*E*+00	−0.197598*E*−01	−0.410961*E*−01	−0.303163*E*+00
7.00	0.300079*E*+00	−0.468020*E*−02	−0.259471*E*−01	−0.302667*E*+00
7.05	0.299938*E*+00	0.103034*E*−01	−0.108414*E*−01	−0.301436*E*+00
7.10	0.299051*E*+00	0.251559*E*−01	0.418443*E*−02	−0.299478*E*+00
7.15	0.297425*E*+00	0.398428*E*−01	0.190945*E*−01	−0.296803*E*+00
7.20	0.295070*E*+00	0.543301*E*−01	0.338531*E*−01	−0.293422*E*+00
7.25	0.291996*E*+00	0.685844*E*−01	0.484252*E*−01	−0.289347*E*+00
7.30	0.288216*E*+00	0.825731*E*−01	0.627766*E*−01	−0.284593*E*+00
7.35	0.283744*E*+00	0.962645*E*−01	0.768736*E*−01	−0.279176*E*+00
7.40	0.278595*E*+00	0.109628*E*+00	0.906836*E*−01	−0.273114*E*+00
7.45	0.272787*E*+00	0.122633*E*+00	0.104175*E*+00	−0.266424*E*+00
7.50	0.266338*E*+00	0.135251*E*+00	0.117316*E*+00	−0.259127*E*+00
7.55	0.259269*E*+00	0.147454*E*+00	0.130078*E*+00	−0.251244*E*+00
7.60	0.251600*E*+00	0.159216*E*+00	0.142431*E*+00	−0.242799*E*+00
7.65	0.243355*E*+00	0.170511*E*+00	0.154349*E*+00	−0.233815*E*+00
7.70	0.234557*E*+00	0.181315*E*+00	0.165804*E*+00	−0.224316*E*+00
7.75	0.225232*E*+00	0.191605*E*+00	0.176772*E*+00	−0.214330*E*+00

(Cont.)

Table H.1 Bessel functions of integer order (continued)

X	J_0	J_1	Y_0	Y_1
7.80	0.215405E+00	0.201359E+00	0.187230E+00	−0.203883E+00
7.85	0.205105E+00	0.210557E+00	0.197153E+00	−0.193003E+00
7.90	0.194359E+00	0.219181E+00	0.206523E+00	−0.181718E+00
7.95	0.183197E+00	0.227214E+00	0.215319E+00	−0.170060E+00
8.00	0.171648E+00	0.234638E+00	0.223524E+00	−0.158057E+00
8.05	0.159743E+00	0.241441E+00	0.231120E+00	−0.145742E+00
8.10	0.147514E+00	0.247609E+00	0.238093E+00	−0.133146E+00
8.15	0.134993E+00	0.253132E+00	0.244430E+00	−0.120300E+00
8.20	0.122212E+00	0.258000E+00	0.250120E+00	−0.107237E+00
8.25	0.109204E+00	0.262205E+00	0.255151E+00	−0.939909E−01
8.30	0.960023E−01	0.265740E+00	0.259516E+00	−0.805939E−01
8.35	0.826409E−01	0.268602E+00	0.263208E+00	−0.670793E−01
8.40	0.691533E−01	0.270787E+00	0.266223E+00	−0.534807E−01
8.45	0.555734E−01	0.272293E+00	0.268556E+00	−0.398314E−01
8.50	0.419352E−01	0.273122E+00	0.270205E+00	−0.261648E−01
8.55	0.282724E−01	0.273275E+00	0.271172E+00	−0.125140E−01
8.60	0.146188E−01	0.272755E+00	0.271458E+00	0.108798E−02
8.65	0.100795E−02	0.271567E+00	0.271065E+00	0.146084E−01
8.70	−0.125270E−01	0.269718E+00	0.269999E+00	0.280150E−01
8.75	−0.259531E−01	0.267217E+00	0.268266E+00	0.412758E−01
8.80	−0.392381E−01	0.264073E+00	0.265874E+00	0.543596E−01
8.85	−0.523499E−01	0.260296E+00	0.262833E+00	0.672355E−01
8.90	−0.652575E−01	0.255901E+00	0.259154E+00	0.798733E−01
8.95	−0.779300E−01	0.250900E+00	0.254850E+00	0.922439E−01
9.00	−0.903378E−01	0.245310E+00	0.249935E+00	0.104318E+00

9.05	−0.102452*E*+00	0.239147*E*+00	0.244424*E*+00	0.116069*E*+00
9.10	−0.114243*E*+00	0.232428*E*+00	0.238334*E*+00	0.127470*E*+00
9.15	−0.125686*E*+00	0.225175*E*+00	0.231683*E*+00	0.138494*E*+00
9.20	−0.136752*E*+00	0.217406*E*+00	0.224491*E*+00	0.149116*E*+00
9.25	−0.147418*E*+00	0.209144*E*+00	0.216778*E*+00	0.159314*E*+00
9.30	−0.157659*E*+00	0.200411*E*+00	0.208567*E*+00	0.169065*E*+00
9.35	−0.167452*E*+00	0.191231*E*+00	0.199880*E*+00	0.178346*E*+00
9.40	−0.176775*E*+00	0.181629*E*+00	0.190740*E*+00	0.187139*E*+00
9.45	−0.185608*E*+00	0.171630*E*+00	0.181174*E*+00	0.195423*E*+00
9.50	−0.193932*E*+00	0.161261*E*+00	0.171207*E*+00	0.203183*E*+00
9.55	−0.201728*E*+00	0.150548*E*+00	0.160865*E*+00	0.210400*E*+00
9.60	−0.208981*E*+00	0.139521*E*+00	0.150176*E*+00	0.217061*E*+00
9.65	−0.215676*E*+00	0.128206*E*+00	0.139168*E*+00	0.223153*E*+00
9.70	−0.221798*E*+00	0.116634*E*+00	0.127870*E*+00	0.228662*E*+00
9.75	−0.227335*E*+00	0.104834*E*+00	0.116312*E*+00	0.233579*E*+00
9.80	−0.232278*E*+00	0.928354*E*−01	0.104522*E*+00	0.237895*E*+00
9.85	−0.236616*E*+00	0.806689*E*−01	0.925323*E*−01	0.241602*E*+00
9.90	−0.240342*E*+00	0.683650*E*−01	0.803723*E*−01	0.244693*E*+00
9.95	−0.243451*E*+00	0.559543*E*−01	0.680732*E*−01	0.247166*E*+00
10.00	−0.245937*E*+00	0.434677*E*−01	0.556660*E*−01	0.249016*E*+00
10.05	−0.247797*E*+00	0.309362*E*−01	0.431819*E*−01	0.250242*E*+00
10.10	−0.249030*E*+00	0.183904*E*−01	0.306521*E*−01	0.250845*E*+00
10.15	−0.249636*E*+00	0.586119*E*−02	0.181077*E*−01	0.250825*E*+00
10.20	−0.249617*E*+00	−0.662092*E*−02	0.557987*E*−02	0.250185*E*+00
10.25	−0.248975*E*+00	−0.190256*E*−01	−0.690065*E*−02	0.248932*E*+00
10.30	−0.247716*E*+00	−0.313230*E*−01	−0.193032*E*−01	0.247069*E*+00
10.35	−0.245845*E*+00	−0.434835*E*−01	−0.315976*E*−01	0.244605*E*+00

(Cont.)

Table H.1 Bessel functions of integer order (continued)

X	J_0	J_1	Y_0	Y_1
10.40	$-0.243371E+00$	$-0.554779E-01$	$-0.437540E-01$	$0.241549E+00$
10.45	$-0.240301E+00$	$-0.672777E-01$	$-0.557429E-01$	$0.237911E+00$
10.50	$-0.236646E+00$	$-0.788550E-01$	$-0.675356E-01$	$0.233702E+00$
10.55	$-0.232419E+00$	$-0.901825E-01$	$-0.791039E-01$	$0.228936E+00$
10.60	$-0.227633E+00$	$-0.101234E+00$	$-0.904203E-01$	$0.223627E+00$
10.65	$-0.222301E+00$	$-0.111982E+00$	$-0.101458E+00$	$0.217790E+00$
10.70	$-0.216440E+00$	$-0.122404E+00$	$-0.112191E+00$	$0.211442E+00$
10.75	$-0.210066E+00$	$-0.132475E+00$	$-0.122594E+00$	$0.204601E+00$
10.80	$-0.203199E+00$	$-0.142171E+00$	$-0.132643E+00$	$0.197286E+00$
10.85	$-0.195856E+00$	$-0.151471E+00$	$-0.142315E+00$	$0.189517E+00$
10.90	$-0.188058E+00$	$-0.160354E+00$	$-0.151588E+00$	$0.181315E+00$
10.95	$-0.179828E+00$	$-0.168799E+00$	$-0.160440E+00$	$0.172702E+00$
11.00	$-0.171186E+00$	$-0.176789E+00$	$-0.168851E+00$	$0.163701E+00$
11.05	$-0.162157E+00$	$-0.184305E+00$	$-0.176804E+00$	$0.154336E+00$
11.10	$-0.152764E+00$	$-0.191332E+00$	$-0.184279E+00$	$0.144632E+00$

Appendix I: Instructions for Computer Programs

COMPUTER programs used in the text are copied on the floppy disk as attached. The programs are written in FORTRAN-77. They can be compiled by the FORTRAN compiler of Watcom System, Inc. The following procedure is for running the programs:

1) Run the computer on the directory where FORTRAN-77 compiler is located; say FORTRAN.

2) Insert the floppy disk to disk drive (A or B).

3) Copy the program to the FORTRAN directory.

4) Type "watfor77 *filename*"

Each program will generate a set of computed results for a plot. Much computer software such as C-plot, sigmaplot, and mathcad, etc., can be used for plotting. Separate software must be used to plot the graphs. To modify the program, edit the files after copying them into the directory.

Type "weditf *filename*"

After the modifications are done, save the changes:

Type "autosave"

The following is the list of programs available on the disk:

1) MSSL.FOR (for Example 2.1)
2) SPACEV.FOR (for Example 2.2)
3) MSLTMSLT.FOR (for Example 3.1 with spherical ground)
4) MSLTOMSL.FOR (for Example 3.1 with flat ground)
5) ELCTPRP.FOR (for Fig. 5.6 spiral orbit of electrical propulsion)
6) TOPS0.FOR (for Fig. 7.11c, nutation of top in Section 7.5)
7) TOPS1.FOR (for Fig. 7.11b, nutation of top in Section 7.5)
8) TOPS2.FOR (for Fig. 7.11a, nutation of top in Section 7.5)
9) RSPNS.FOR (for Example 8.14, response spectrum)
10) STBLTYI.FOR (for case $E = 1$ in Example 9.18)
11) STBLTY2.FOR (for case $E = 2$ in Example 9.18)
12) STBLTY3.FOR (for case $E = 3$ in Example 9.18)
13) BESSJ0.FOR (for Bessel function J_0 in Appendix H)
14) BESSJ1.FOR (for Bessel function J_1 in Appendix H)
15) BESSY0.FOR (for Bessel function Y_0 in Appendix H)
16) BESSY1.FOR (for Bessel function Y_1 in Appendix H)
17) BSFJY.FOR (for Table of Bessel Functions in Appendix H)

Appendix J: Further Reading

Textbooks on Advanced Dynamics—Graduate Level

Barlett, J., *Classical and Modern Mechanics,* Univ. of Alabama Press, Huntsville, AL, 1975.

D'Souza, A. F., and Garg, V. K., *Advanced Dynamics,* Prentice-Hall, Englewood Cliffs, NJ, 1984.

Goldstein, H., *Classical Mechanics,* 2nd ed., Addison-Wesley, Reading, MA, 1980.

Greenwood, D. T., *Principles of Dynamics,* Prentice-Hall, Englewood Cliffs, NJ, 1988.

Groesberg, S., *Advanced Mechanics,* Wiley, New York, 1971.

Moorse, E. N., *Theoretical Mechanics,* Wiley, New York, 1983.

Textbooks on Mechanics—Undergraduate Level

Beer, F. P., and Johnston, E. R., *Vector Mechanics for Engineers,* 3rd ed., McGraw-Hill, New York, 1977.

Hibbler, R. C., *Engineering Mechanics,* Macmillan, New York, 1986.

Martin, G. H., *Kinematics and Dynamics of Machines,* 2nd ed., McGraw-Hill, New York, 1982.

Meriam, J. L., and Kraige, L. G., *Engineering Mechanics,* Wiley, New York, 1987.

Shames, I. H., *Engineering Mechanics,* 3rd ed., Prentice-Hall, Englewood Cliffs, NJ, 1980.

Synge, J. L., and Griffith, B. A., *Principles of Mechanics,* 3rd ed., McGraw-Hill, New York, 1959.

Textbooks on Mathematics Relevant to Mechanics

Jeffreys, H., and Jeffreys, B. S., *Methods of Mathematical Physics,* 3rd ed., Cambridge Univ. Press, London, 1956.

Hildebrand, F. B., *Advanced Calculus for Applications,* Prentice-Hall, Englewood Cliffs, NJ, 1976.

Sokolnikoff, I. S., and Redheffer, R. M., *Mathematics of Physics and Modern Engineering,* 2nd ed., McGraw-Hill, New York, 1966.

Other Books of Interest

Chobotov, V. A., *Orbital Mechanics,* AIAA Education Series, AIAA, Washington, DC, 1991.

Hughes, P. C., *Spacecraft Attitude Dynamics,* Wiley, New York, 1986.

Regan, F. J., and Anandakrishnan, S. M., *Dynamics of Atmospheric Re-entry,* AIAA Education Series, AIAA, Washington, DC, 1993.

Shabana, A. A., *Theory of Vibration,* Springer-Verlag, New York, 1991.

Thomson, W. T., *Theory of Vibration with Applications,* 3rd ed., Prentice-Hall, Englewood Cliffs, NJ, 1988.

Willems, P. Y. (ed.), Gyrodynamics, Euromech 38 Colloquium, Springer-Verlag, Berlin, 1974.

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