Density profile of a hole at the fractional Hall edge

Gun Sang Jeon and J. K. Jain
Department of Physics, 104 Davey Laboratory, The Pennsylvania State University, University Park, Pennsylvania 16802, USA
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The density profile of a hole excitation is investigated when it is located at the edge of the fractional quantum Hall state at filling factor 1/3. Of interest is the issue how the hole profile decays at long distances, both along the edge and into the bulk. We find that the hole is exponentially localized in both directions. Its localization length along the edge depends on the range of the interelectron interaction, increasing roughly by a factor of two in going from a short-range interaction to the long-range Coulomb interaction.

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There has been much interest in the physics describing the edge of a fractional quantum Hall effect (FQHE) state.[1] Because of a gap in the bulk, the only low-energy excitations of a FQHE system occur at the edge, and it is believed that their dynamics can be mapped into that of a one-dimensional Tomonaga-Luttinger liquid.[2] There has been much experimental work toward determining the value of the exponent that describes the long-distance behavior of this liquid, through the *I-V* curve for tunneling of an electron into the edge.[1,3,4] The measured exponent appears to be in disagreement with its theoretical interpretation as a topological quantum number.[2] A number of recent theoretical works[5–11] have addressed the nonuniversality of the exponent.

An interesting feature noted in Refs. 5, 7, and 12 is that the long-range Coulomb interaction induces density oscillations at the edge that decay slowly, apparently with a power law, as one moves into the interior. Other theoretical works[9,10] have indicated the possibility of edge reconstruction. That brings up the question that we wish to address in this article: How one-dimensional is the edge? The slow decay of density oscillations may suggest that the edge physics is not entirely decoupled from the bulk physics. However, these density oscillations are a part of the ground state. The important question is whether *excitations* at the edge are confined to the edge, or extend into the bulk. If the excitations at the edge are not confined to the edge, that would imply that the physics of the edge is, strictly speaking, not one-dimensional. Our modest aim in this paper is to investigate this issue using an approach[5,7,12] that does not allow for edge reconstruction.

For filling factor unity, if one neglects spin and higher Landau levels, it is clear that the low-energy excitations are confined to the edge and have a mapping into a system of one-dimensional fermions.[13] Neglect of higher Landau levels is possibly justified in the limit of extremely large magnetic fields, when the cyclotron energy overwhelms the interaction energy between electrons. (If one takes electrons to be noninteracting, then, of course, higher Landau levels can be neglected at any magnetic field.) However, such a mapping has not been established rigorously for the FQHE states. Indeed, the slow decay of density oscillations in the ground state is an indication of the fact that the bulk is not as inert in FQHE as it is in the integral quantum Hall effect (IQHE). The reason is that the mixing between *composite-fermion* (CF) quasi-Landau-levels cannot be neglected, due to the fact that the effective cyclotron energy for the composite fermions is governed by the same energy scale as the interaction strength.

Here, we study the profile of a hole at the edge, obtained by annihilating an electron at the edge. (The hole is not to be confused with the fractionally charged quasihole.) The hole is not a low-energy excitation, but is relevant for tunneling, as this is precisely the state produced when an electron tunnels out of the FQHE edge. (A determination of the tunneling conductance requires the spectral resolution of this state, which is beyond the scope of this work.) Our main conclusion is that even though the density of a hole at the edge has a much more complicated structure at short distances than that of a hole in the interior, the envelope of the density profile decays exponentially. We confirm this for wave functions at different levels of approximation, as well as for the exact wave function for small systems. The extension of a hole at the edge into the bulk is approximately restricted within four magnetic lengths.

For a given ground state wave function $\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N)$, the hole at $\mathbf{R}$ is given by

$$|\Psi^h_{\mathbf{R}}\rangle = \hat{\Psi}_{\mathbf{R}}|\Psi\rangle, \tag{1}$$

where $\hat{\Psi}_{\mathbf{R}} = \Sigma_\alpha \eta_\alpha(\mathbf{R}) c_\alpha$ is the standard annihilation operator [$\eta_\alpha(\mathbf{r})$ are single-particle eigenstates, and c_α destroys an electron in the state α], and $|\Psi\rangle$ denotes the wave function in the second quantized form. The wave function for the hole at $\mathbf{R}$, obtained by the application of $\hat{\Psi}_{\mathbf{R}}$ to the ground state, is given by

$$\Psi^h_{\mathbf{R}} = \Psi(\mathbf{R}, \mathbf{r}_2, \ldots, \mathbf{r}_N). \tag{2}$$

We will compute the density of a hole (measured relative to the ground state density) situated at the edge.

We will work in the disk geometry (the spherical geometry, often used for the study of the FQHE states, is not useful for our present purpose for lack of edges). The calculation requires a sufficiently accurate approximation for the ground-state wave function, Ψ. The exact ground-state wave function for the Coulomb interaction, Ψ^{exact}, can be evaluated from the Hamiltonian of a second-quantized form

$$\mathcal{H} = \frac{1}{2}\sum_{r,s,t,u} \langle r,s|V|t,u\rangle a_r^\dagger a_s^\dagger a_u a_t, \tag{3}$$

where the operator $a_r^\dagger$ (a_r) creates (annihilates) an electron at state $|r\rangle$ with angular momentum r and V is the Coulomb interaction. For the Coulomb interaction matrix element we used the expression derived by Goldman and Tsiper[6,12,14]

$$\langle s+r,t|V|s,t+r\rangle = \sqrt{\frac{(s+r)!\,(t+r)!}{s!\,t!}} \times \frac{\Gamma\left(r+s+t+\frac{3}{2}\right)}{\pi 2^{r+s+t+2}}[A^r_{st}B^r_{ts} + B^r_{st}A^r_{ts}], \tag{4}$$

with

$$A^r_{st} = \sum_{i=0}^{s}\binom{s}{i}\frac{\Gamma\left(\frac{1}{2}+i\right)\Gamma\left(\frac{1}{2}+r+i\right)}{(r+i)!\,\Gamma\left(\frac{3}{2}+r+t+i\right)}, \tag{5}$$

$$B^r_{st} = \sum_{i=0}^{s}\binom{s}{i}\frac{\Gamma\left(\frac{1}{2}+i\right)\Gamma\left(\frac{1}{2}+r+i\right)}{(r+i)!\,\Gamma\left(\frac{3}{2}+r+t+i\right)}\left(\frac{1}{2}+r+2i\right). \tag{6}$$

The above expressions are more stable for numerical work than those used previously[15,16] because they consist of finite sums of *positive* terms. In order to obtain the ground state we have employed a modified Lanczos algorithm,[17] which has allowed computation at filling factor $\nu=1/3$ for up to $N=10$.

For larger systems, exact diagonalization is not possible, and one must resort to variational wave functions.[18,19] We will investigate the structure of the edge hole in the context of $\nu=1/3$, where a good approximation to the exact ground state is given by Laughlin's wave function.[18] To obtain a better representation, we will use a method called "CF diagonalization,"[5,7,20] which refers to a diagonalization of the full Hamiltonian (which is simply the Coulomb interaction in the lowest-Landau-level subspace) in a relatively small strongly correlated basis constructed with the help of the CF theory. The advantage of this method is that, because of the smaller dimension of the basis, it can be used when exact diagonalization is not possible. It has been shown to be very accurate, and can be improved by successively increasing the dimension of the correlated basis, which, in turn, involves increasing the amount of mixing between CF quasi-Landau-levels.[20] The disadvantage is that the calculation of Hamiltonian matrix elements is more involved than that for simple Slater determinant basis functions, and requires extensive Monte Carlo evaluation. The method, which involves several steps, has been described in the literature[5,7,21] and will not be repeated here in detail. Here we only provide a brief outline.

The CF theory[16,19,21,22] maps strongly interacting electrons at angular momentum L into weakly interacting electrons at $L^* \equiv L - pN(N-1)$. A correlated basis $\{\Psi^L_\alpha\}$ for the low-energy states of interacting electrons at L can be constructed from the trivial, orthonormal Slater determinant basis for noninteracting electrons at L^*, denoted by $\{\Phi^{L^*}_\alpha\}$, in the following manner:

$$\Psi^L_\alpha = \mathcal{P}\prod_{j<k}(z_j - z_k)^{2p}\Phi^{L^*}_\alpha. \tag{7}$$

Here, $z_j = x_j - iy_j$ denotes the position of the jth electron, $2p$ is the vorticity of composite fermions, and $\mathcal{P}$ indicates projection into the lowest Landau level. The symbol $\alpha = 1,2,\ldots,D^*$ labels the D^* Slater determinants included in the study. The ground state is then obtained by a diagonalization of the full Hamiltonian in the space defined by $\{\Psi^L_\alpha\}$. Different choices of the starting basis $\{\Phi^{L^*}_\alpha\}$ produce different approximations for the ground state at L.

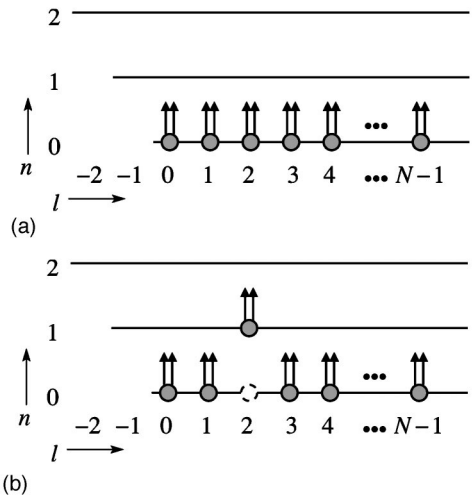

FIG. 1. The zeroth-order wave function Ψ^0 is schematically depicted in (a). Ψ' is obtained by a diagonalization of the full Hamiltonian in the composite-fermion basis states of the type shown schematically in (a) and (b).

At $\nu=1/3$ we have $L=L^*+N(N-1)$ and $L^*=N(N-1)/2$. We have considered three wave functions for the ground state. (i) Ψ^0 denotes the zeroth-order wave function, obtained when we keep only the ground state at L^*. It is schematically shown in Fig. 1(a). The wave function Ψ^0 is identical to Laughlin's wave function. (ii) Ψ' is an improved approximation, obtained by allowing mixing with a single particle-hole pair of composite fermions; this was considered in Ref. 5. Ψ' is obtained by a diagonalization of the full Hamiltonian in the space defined by basis functions of the type shown in Figs. 1(a) and 1(b). (iii) Finally, for small systems, we also have available the exact wave function, Ψ^{exact}, which will be evaluated for the Coulomb interaction. The length will be measured in units of the magnetic length $l \equiv \sqrt{\hbar c/eB}$ throughout the paper.

The above three wave functions can be interpreted as different levels of approximation to the Coulomb ground state of the many-electron system. The accuracy increases in the order $\Psi^0 \to \Psi' \to \Psi^{\text{exact}}$. Alternatively, one can interpret the results in terms of the range of the interaction. Ψ^0 is known to be the exact ground state for a short-range interaction,[23] while Ψ^{exact} is, by definition, exact for the *full* Coulomb in-

teraction. Because Ψ' is somewhere in between the two, it can be identified with the ground state of some (unknown) interaction whose effective range is between the short-range interaction and the Coulomb interaction. Accordingly, the change of the properties in the sequence $\Psi^0 \to \Psi' \to \Psi^{\text{exact}}$ can be regarded as a reflection of the variation in the effective range of the interaction between electrons.

The electron density for the state containing a hole at $\mathbf{R}$ is given by

$$\rho_{\mathbf{R}}(\mathbf{r}) = \frac{\int d^2\mathbf{r}_3 \cdots d^2\mathbf{r}_N |\Psi(\mathbf{R},\mathbf{r},\mathbf{r}_3,\cdots,\mathbf{r}_N)|^2}{\int d^2\mathbf{r}_2 \cdots d^2\mathbf{r}_N |\Psi(\mathbf{R},\mathbf{r}_2,\cdots,\mathbf{r}_N)|^2}, \tag{8}$$

which we evaluate by the Monte Carlo method. It is noted that this is identical to the pair correlation function $g(\mathbf{R},\mathbf{r})$

$$g(\mathbf{R},\mathbf{r}) \equiv \langle \delta(\mathbf{r}_1 - \mathbf{R})\,\delta(\mathbf{r}_2 - \mathbf{r})\rangle \tag{9}$$

apart from an overall normalization factor. We define the density of the hole as

$$\rho^h_{\mathbf{R}}(\mathbf{r}) = -[\rho_{\mathbf{R}}(\mathbf{r}) - \rho^0(\mathbf{r})], \tag{10}$$

which is also the excess charge density in units of the electron charge. Here ρ^0 is the electron density for the ground state, given by

$$\rho^0(\mathbf{r}) = \frac{\int d^2\mathbf{r}_2 \cdots d^2\mathbf{r}_N |\Psi(\mathbf{r},\mathbf{r}_2,\mathbf{r}_3,\cdots,\mathbf{r}_N)|^2}{\int d^2\mathbf{r}_1 \cdots d^2\mathbf{r}_N |\Psi(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)|^2}. \tag{11}$$

As noted in Refs. 5 and 12, the exact density shows strong radial oscillations, which are accurately captured by Ψ' but not by Ψ_0.

We now place one hole at $\mathbf{R}=6\hat{\mathbf{x}}$, at the maximum in the ground-state density near the edge. The hole density profile has a peak at $\mathbf{R}$, which has a radius of $\approx 4l$ both in the radial and angular directions. Along the circumference [Fig. 2(a)], the hole profile exhibits oscillations attenuating with distance. The amplitude of the oscillations increases with improving accuracy. For Ψ^0, we can hardly identify any oscillations beyond the second valley. Ψ' captures the long-range nature of the oscillations, but underestimates the amplitude. The period of the oscillations is obtained correctly by all three wave functions. For the radial direction, we find that Ψ' provides an almost exact account of the hole density, as noted by the fact that the solid and dashed lines are indistinguishable in Fig. 2(b).

Now we come to the question of whether the edge excitations are confined to the edge of a FQHE system. Figure 2(b) shows that the hole is well localized near the edge. To study the dependence on the number of particles, we display in Fig. 3 the density profile of the hole for $N=10$ and $N=20$ particles, when it is located approximately at the maximum ground state density at the edge. (We have chosen $R=6$ for $N=10$ and $R=9$ for $N=20$). Since the exact wave function Ψ^{exact} is not available for $N=20$, we have used the wave function Ψ', which was seen to give a practically exact

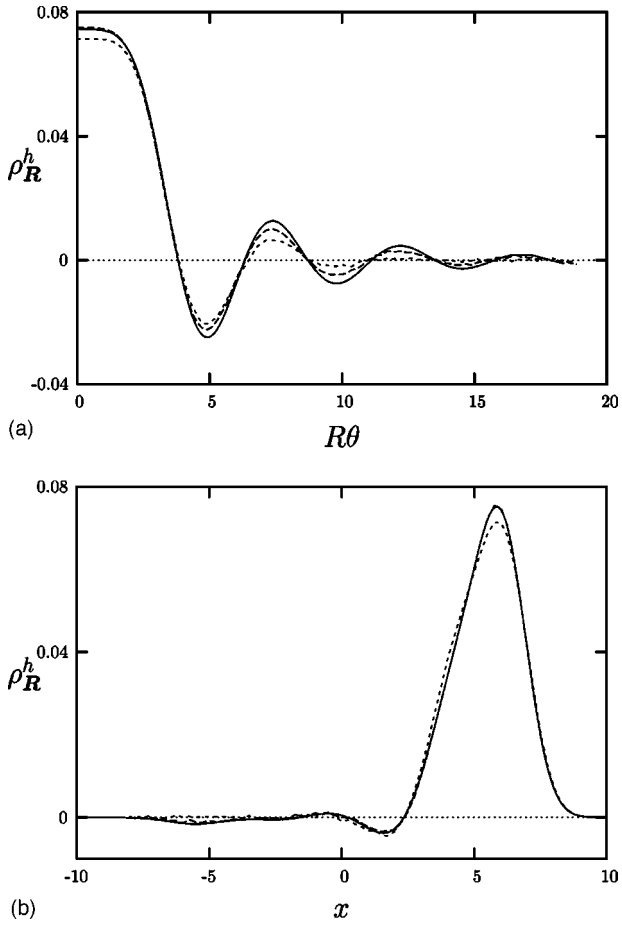

FIG. 2. The density profile of the hole excitation for $\mathbf{R}=6\hat{\mathbf{x}}$ (a) in the angular direction; (b) in the radial direction. The density is quoted in units of the inverse square of magnetic length l^{-2}. [In this unit, the bulk density for $\nu=1/3$ is $1/(6\pi)\approx 0.053$.] The solid, the dashed, and the dotted lines correspond to Ψ^{exact}, Ψ', and Ψ^0, respectively. The density of the ground state as a function of r in Refs. 5 and 12 has a peak at $r\approx 6l$, beyond which it decreases exponentially.

result for ten particles. Figure 3 indicates that the hole density profile remains confined to the edge in the thermodynamic limit. [The size of the hole has increased very slightly in going from $N=10$ to $N=20$, but the change is very small compared with the increase in the disk size ($\approx 3l$), and is unrelated to the extension of a hole into the bulk.]

What is the behavior in the azimuthal direction along the edge of the disk? The equal-time Green's function along the FQHE edge is known to exhibit a power-law decay. Some microscopic studies[5–7] have suggested that the exponent depends on the nature of the electron-electron interaction. A quick inspection of Fig. 2(a) tells us that the hole density profile is also affected by the interaction: As we go from a short-range interaction to the long-range Coulomb interaction ($\Psi^0 \to \Psi' \to \Psi^{\text{exact}}$), the oscillations become more prominent.

For a quantitative analysis, we have investigated the behavior of the oscillation amplitude as a function of the distance along the edge. The amplitude of the oscillation at each extremum can be estimated by the difference $\Delta\rho^h_{\mathbf{R}}$ between

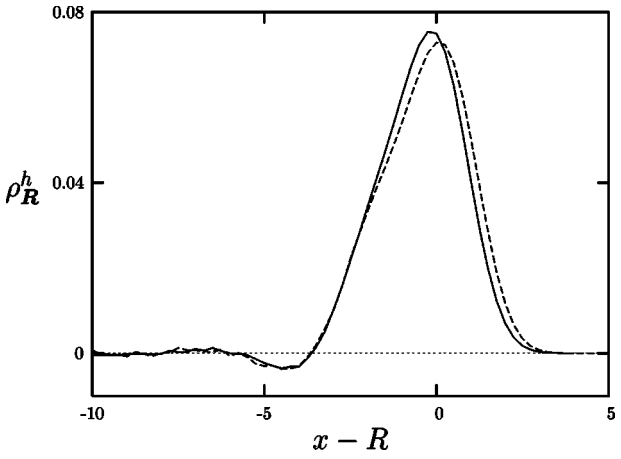

FIG. 3. The density profile in the radial direction for the hole excitation at $\mathbf{R}=6\hat{\mathbf{x}}$ for $N=10$ (solid line) and at $\mathbf{R}=9\hat{\mathbf{x}}$ for $N=20$ (dashed line). The wave function Ψ' has been used for the calculation, and the origin has been shifted to the hole position for comparison.

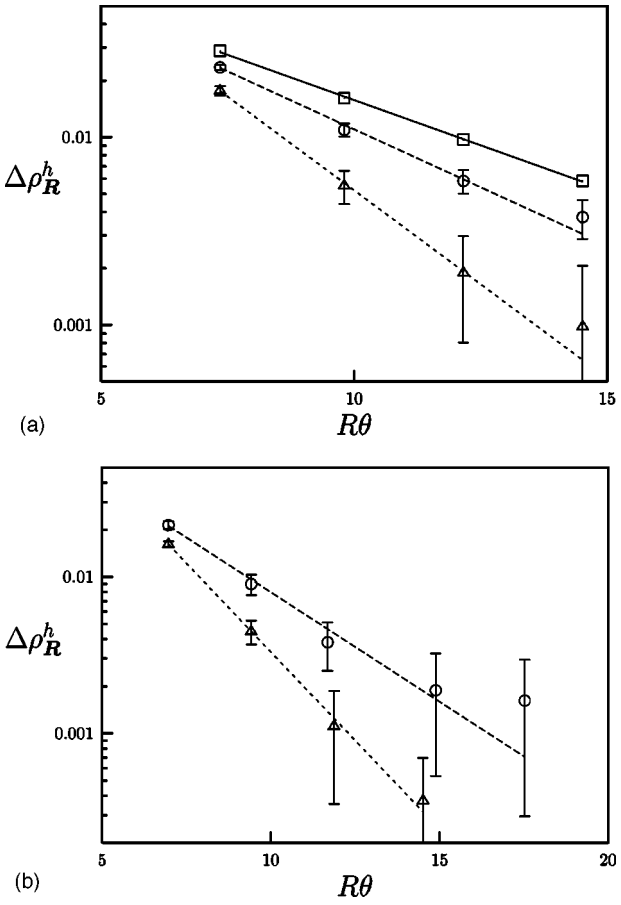

FIG. 4. Semilogarithmic plot of the oscillation amplitude of the hole density profile as a function of the distance along the disk edge for (a) $N=10$ and $R=6$; (b) $N=20$ and $R=9$. The data for Ψ^{exact}, Ψ', and Ψ^0 are denoted by squares, circles and triangles, respectively. The best fits to the equation $\Delta\rho^h_{\mathbf{R}}(R\theta)=A\exp(-R\theta/\xi)$ are also shown.

TABLE I. The localization length ξ of the hole density profile along the edge. We put one hole at $\mathbf{R}=6\hat{\mathbf{x}}$ for $N=10$ and $\mathbf{R}=9\hat{\mathbf{x}}$ for $N=20$. Even for the $N=10$ system, all the estimated localization lengths are far smaller than half of the circumference considered ($\pi R\approx 18.8$), which guarantees the validity of our estimation. The exact value for $N=20$ is unavailable due to the huge size of the Hilbert space.

N	Ψ^0	Ψ'	Ψ^{exact}
10	2.2	3.5	4.5
20	1.9	3.1	—

the upper and the lower envelope functions of the hole density profile. At each minimum we have approximated the value of the upper envelope function by a linear interpolation of two adjacent maxima, and vice versa. Even though the decay of the oscillations depends on the form of the wave function (and hence on the form of the interaction), the hole appears, quite generally, to be exponentially localized along the edge. By fitting $\Delta\rho^h_{\mathbf{R}}$ of each wave function to an exponential function, we obtain the localization length ξ in the angular direction. [The fitting curves are also plotted by lines

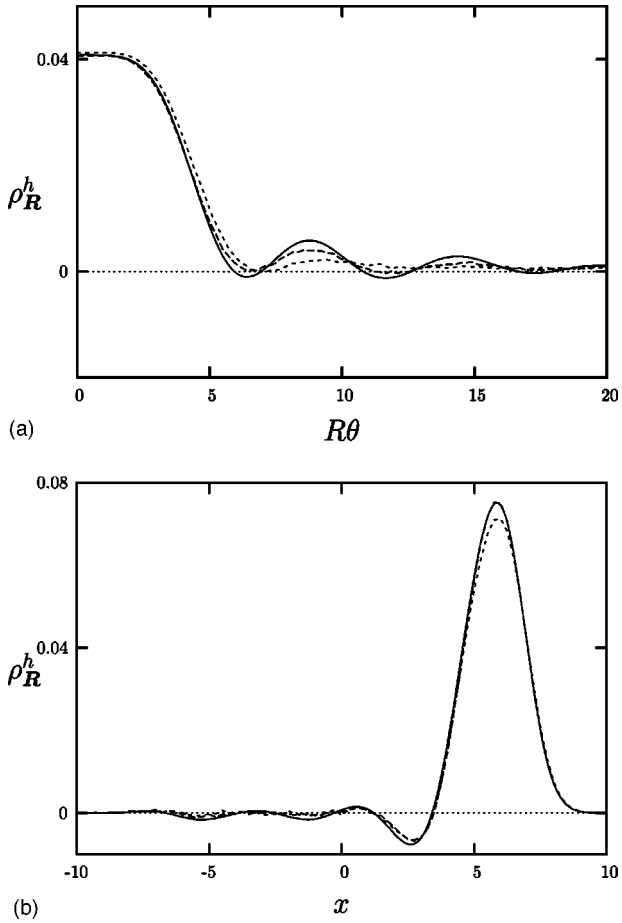

FIG. 5. The density profile of the hole excitation for $\mathbf{R}=7\hat{\mathbf{x}}$ (a) in the angular direction; (b) in the radial direction. There are 10 electrons in the system, and the solid, the dashed, and the dotted lines correspond to Ψ^{exact}, Ψ', and Ψ^0, respectively.

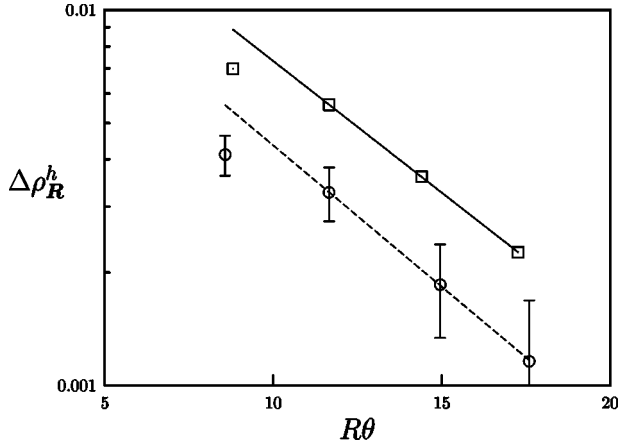

FIG. 6. Semilogarithmic plot of the oscillation amplitude of the hole density profile as a function of the distance along the disk edge for $N=10$ and $R=7$. The data for Ψ^{exact} and Ψ' are denoted by squares and circles, respectively. The best fits to the equation $\Delta\rho^h_{\mathbf{R}}(R\theta)=A\exp(-R\theta/\xi)$ are also shown. (The first point is neglected for the fitting.)

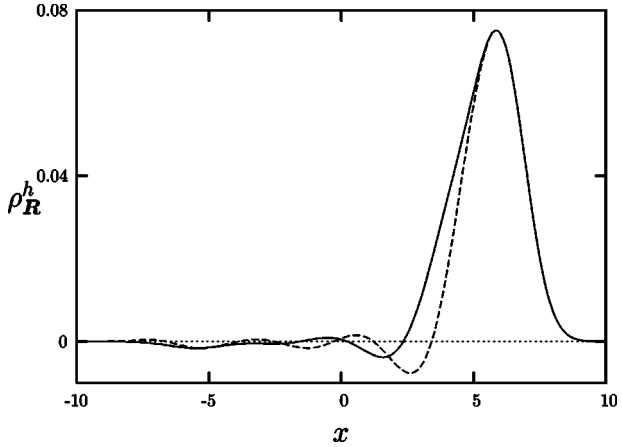

FIG. 7. The density profile for the hole excitation in the radial direction for $\mathbf{R}=6\hat{\mathbf{x}}$ (solid line) and $\mathbf{R}=7\hat{\mathbf{x}}$ (dashed line). There are 10 electrons in the system.

in Fig. 4(a).] The localization lengths for different wave functions are summarized in Table I. The long-range Coulomb interaction enhances the localization length ξ to 4.5 magnetic lengths, which is approximately twice the localization length for a short-range interaction. The exponential localization of the hole density as well as the interaction dependence of the localization length has been confirmed for a larger system ($N=20$) as shown in Fig. 4(b) and Table I. We believe that the conclusions are valid in the thermodynamic limit.

So far we have examined a hole placed at the maximum in the ground state density near the edge. One may ask if the behavior is sensitive to the exact position of the hole. To investigate this question, we also study a hole at $\mathbf{R}=7\hat{\mathbf{x}}$ (for $N=10$), where the density of the ground state is $\approx 0.55\rho^0(r=6)$. Figure 5(a) shows that oscillations in the angular direction are also present for $R=7$, again most prominent for the long-range interaction. The only difference is that the oscillations are not symmetric about zero. Proceeding as before, we find that the hole profile far from its center decays exponentially for $\mathbf{R}=7\hat{\mathbf{x}}$ as well (see Fig. 6), although with slightly different localization lengths ($\xi\approx 5.8$ for Ψ' and $\xi \approx 6.2$ for Ψ^{exact}). For Ψ^0, the oscillations are too weak for a meaningful analysis. The hole density is well localized in the radial direction also for $R=7$.

A comparison of the density profiles for holes located at $R=6$ and $R=7$ in Fig. 7 shows that the maximum in the hole density occurs at $r=6$ independent of the hole location at the edge, and the density profiles are identical in the outer region. The reason is that in the close vicinity of the hole position $\mathbf{R}$, the density $\rho_{\mathbf{R}}$ is essentially zero, and consequently, the hole density is essentially the same as the ground state density. In particular, the maximum of the hole density matches the maximum in the ground state density.

In summary, we have examined the density profile of a hole excitation at the edge of the fractional quantum Hall state at filling factor $\nu=1/3$. It has been found to be exponentially localized both in the angular and radial directions. The range of the electron-electron interaction affects the localization length along the edge, which is enhanced roughly by a factor of two as the interaction is tuned from a short-range interaction to the long-range Coulomb interaction. In the radial direction, the hole density is confined roughly within four magnetic lengths.

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